CALCULUS 3, SPRING 2010

SECTION 9:30-10:20

FINAL SCALE

578-650	A
495-577	B
397-494	C
320-396	D
0-319	F

CALCULUS 3 EXAM

SPRING 2010

NAME________________

REC. INSTR.________________

REC. TIME________________

TO RECEIVE CREDIT YOU MUST SHOW YOUR WORK

(25) 1. An object is moving in 3-space according to the parametric equations $x = t^2$, $y = 2t$, $z = 2t^2$ where t is the time.

Find, as functions of t,

3 a) position vector $\mathbf{r} = t^2\vec{i} + 2t\vec{j} + 2t^2\vec{k}$

3 b) velocity vector $\mathbf{v} = 2t\vec{i} + 2\vec{j} + 4t\vec{k}$

3 c) acceleration vector $\mathbf{a} = 2\vec{i} + 4\vec{k}$

4 d) speed $ds/dt = |\vec{v}| = \sqrt{4t^2+4+16t^2} = \sqrt{4+20t^2} = 2\sqrt{1+5t^2}$

4 e) tangential component of acceleration $a_T = \frac{d}{dt}(\text{SPEED})$

$$= \frac{1}{2}(4+20t^2)^{-\frac{1}{2}}(40t) = \frac{20t}{\sqrt{4+20t^2}} = \frac{10t}{\sqrt{1+5t^2}}$$

4 f) curvature K =

$$\vec{v}\times\vec{a} = \begin{vmatrix} i & j & k \\ 2t & 2 & 4t \\ 2 & 0 & 4 \end{vmatrix} = 8\vec{i} - 0\vec{j} - 4\vec{k}$$

$$K = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{64+16}}{(4+20t^2)^{3/2}} = \frac{\sqrt{80}}{(4+20t^2)^{3/2}} = \frac{\sqrt{5}}{2(1+5t^2)^{3/2}}$$

4 g) normal component of acceleration a_N =

$$= K|\vec{v}|^2 = \sqrt{\frac{80}{4+20t^2}} = \frac{2\sqrt{5}}{(1+5t^2)^{1/2}}$$

NAME________________

(20) 2. Let $f(x,y,z) = x^2z^3 + y^3z$

5 a) Find the gradient vector field of f

$$\nabla f = 2xz^3\vec{i} + 3y^2z\vec{j} + (3x^2z^2 + y^3)\vec{k}$$

b) Find the directional derivative of f at (x,y,z) = (-2,2,-1) in the direction of the vector 2i + 3j + **k**

$$|2\vec{i} + 3\vec{j} + \vec{k}| = \sqrt{4+9+1} = \sqrt{14}$$

so $\vec{u} = \frac{2}{\sqrt{14}}\vec{i} + \frac{3}{\sqrt{14}}\vec{j} + \frac{1}{\sqrt{14}}\vec{k}$ 2

$$\nabla f(-2,2,-1) = 4\vec{i} - 12\vec{j} + (12+8)\vec{k} = 4\vec{i} - 12\vec{j} + 20\vec{k}$$ 2

6 $$D_{\vec{u}} f(-2,2,-1) = \frac{8}{\sqrt{14}} - \frac{36}{\sqrt{14}} + \frac{20}{\sqrt{14}} = -\frac{8}{\sqrt{14}} \simeq -2.138$$

3 c) find the value of the largest directional derivative for f at the point (-2,2,-1)

$$|\nabla f(-2,2,-1)| = \sqrt{16+144+400} = \sqrt{560} \simeq 23.66$$

2 d) find a vector which points in the direction that gives the largest directional derivative that you found in c)

$$\nabla f(-2,2,-1) = 4\vec{i} - 12\vec{j} + 20\vec{k}$$

(10) 3. Find the equation of the tangent plane to the surface $x^2z^3 + y^3z + 12 = 0$ at the point (-2,2,-1)

from 2., $\langle 4,-12,20\rangle$ is a perp. vector 4

so

$$4(x+2) - 12(y-2) + 20(z+1) = 0$$ 6

NAME________________

(25) 4. Let $f(x,y) = 3x^2y - 2x^3 + y^3 - 24y$. Find all the critical points for f and then apply the 2nd partials test to each critical point to determine its nature.

$\boxed{f_x = 6xy - 6x^2 = 0, \quad f_y = 3x^2 + 3y^2 - 24 = 0}$ (6)

$(6x)(y-x) = 0$

So $x = 0$ or $y = x$

$x = 0$ gives $3y^2 = 24$, $y^2 = 8$, $y = \pm\sqrt{8}$

$y = x$ gives $6x^2 = 24$

$x^2 = 4$, $x = \pm 2$

Thus $\boxed{\text{CP's are } (x,y) = (0,\sqrt{8}), (0,-\sqrt{8}), (2,2), (-2,-2)}$ (4)

$f_{xx} = 6y - 12x$, $f_{yy} = 6y$, $f_{xy} = 6x$

$\boxed{D = (6y - 12x)(6y) - 36x^2}$ (5)

$D(0,\sqrt{8}) = (6\sqrt{8})(6\sqrt{8}) > 0$

Since also $f_{yy}(0,\sqrt{8}) = 6\sqrt{8} > 0$ local min } (3)

$D(0,-\sqrt{8}) = (-6\sqrt{8})(-6\sqrt{8}) > 0$

$f_{yy}(0,-\sqrt{8}) = -6\sqrt{8} < 0$ local max } (3)

When $y = x$, $D = -36x^2 - 36x^2 = -72x^2$

so $(2,2)$ and $(-2,-2)$ are saddle points } (4)

NAME___________

(15) 5. Given the points P(1,2,-1), Q(2,1,3) and R(1,1,1) in xyz-space.

a) find the equation for the plane containing the points P,Q and R

$\overrightarrow{PQ} = \langle 1,-1,4 \rangle, \quad \overrightarrow{PR} = \langle 0,-1,2 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 4 \\ 0 & -1 & 2 \end{vmatrix} = 2\vec{i} - 2\vec{j} - \vec{k} \quad \Big\} \; 5$$

So $2(x-1) - 2(y-1) - (z-1) = 0$

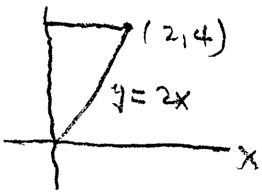
$2x - 2y - z + 1 = 0 \quad \Big\} \; 5$

b) find the area of the triangle having P,Q and R as its vertices

$$\text{area} = \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = \frac{1}{2}\sqrt{4+4+1} = \frac{3}{2} \quad \Big\} \; 5$$

NAME________________

(15) 6. A mass distribution occupies the 3D-region in the 1st octant which is under z = x and above the region in the xy-plane that is enclosed by y = 2x, y = 4 and x = 0 . The mass density function is $\delta = 4xyz$ units of mass/unit volume. Find the total mass in the region.



$$\text{Mass} = \iiint_E 4xyz \, dV \quad \Big\} \; 4$$

$$= \int_0^2 \int_{2x}^4 \int_0^x 4xyz \, dz \, dy \, dx \quad \Big\} \; 6$$

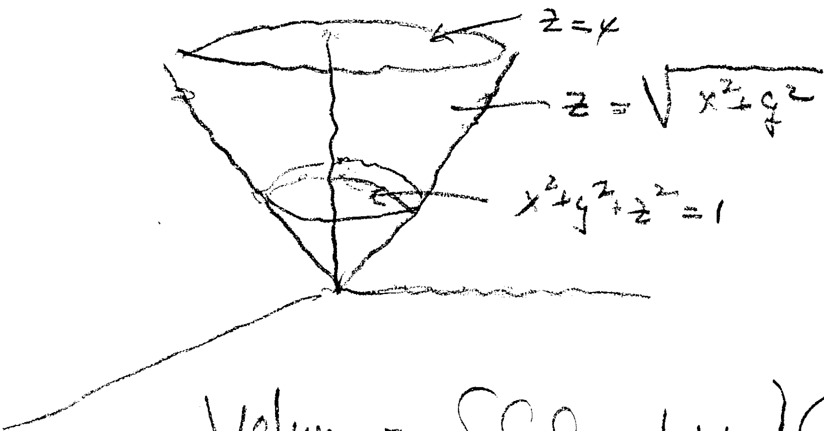
$$= \int_0^2 \int_{2x}^4 2xyz^2 \Big|_{z=0}^{z=x} dy\,dx = \int_0^2 \int_{2x}^4 2x^3 y \, dy\,dx$$

$$= \int_0^2 x^3 y^2 \Big|_{y=2x}^{y=4} dx = \int_0^2 (16x^3 - 4x^5)\,dx$$

$$= 4x^4 - \frac{4}{6}x^6 \Big|_0^2 = 64 - \frac{2}{3}(64) = \frac{64}{3} \text{ units of mass} \quad \Big\} \; 5$$

NAME____________

(15) 7. Use a triple integral in spherical coordinates to calculate the volume of the 3D-region which is enclosed by the surfaces $z = (x^2 + y^2)^{\frac{1}{2}}$, $z = 4$, and $x^2 + y^2 + z^2 = 1$.



$$\text{Volume} = \left.\iiint_E 1\,dV \right\} 4$$

$$= \boxed{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\frac{4}{\cos\phi}} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta} \; 6 = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left.\frac{\rho^3}{3}\right|_1^{\frac{4}{\cos\phi}} \sin\phi \, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left[\frac{64}{3}(\cos\phi)^{-3}\sin\phi - \frac{1}{3}\sin\phi\right] d\phi\, d\theta$$

$$= \int_0^{2\pi} \frac{64}{3}\left.\frac{(\cos\phi)^{-2}}{2}\right|_0^{\frac{\pi}{4}} + \frac{1}{3}\cos\phi \Big|_0^{\frac{\pi}{4}} \, d\theta$$

$$= \int_0^{2\pi} \left[\left(\frac{64}{3}\right)\left(1 - \frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{\sqrt{2}} - 1\right)\right] d\theta$$

$$= \left(\frac{32}{3} - \frac{1}{3} + \frac{1}{3\sqrt{2}}\right) 2\pi$$

$$= \frac{1}{3}\left(31 + \frac{1}{\sqrt{2}}\right) 2\pi \simeq (10.569)(2\pi) = 66.407$$

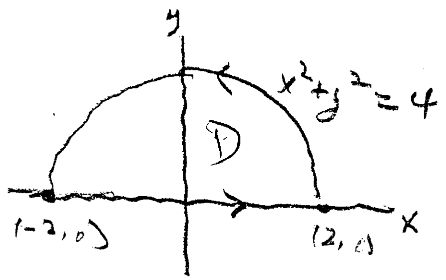
5

NAME____________

(15) 8. An object moves in the xy-plane from (-2,0) to (2,0) along the x-axis and then along the top half of the circle $x^2 + y^2 = 4$ back to (-2,0). All the while it is acted upon by the force field

$$\mathbf{F} = (xy^2 + x^2 - x^2y)\mathbf{i} + (xy^2 + x^2y)\mathbf{j} .$$

Use Green's theorem to calculate the work done by **F** .



$$\text{Work} = \int_C (xy^2 + x^2 - x^2y)\,dx + (xy^2 + x^2y)\,dy \quad \Big\} \; 4$$

$$= \iint_D [(y^2 + 2xy) - (2xy - x^2)]\,dA$$

$$= \iint_D (x^2 + y^2)\,dA \quad \Big\} \; 6$$

$$= \int_0^{\pi}\int_0^{2} r^2 \cdot r\,dr\,d\theta = \int_0^{\pi} \frac{r^4}{4}\Big|_0^2 \,d\theta$$

$$= \int_0^{\pi} 4\,d\theta = 4\pi \quad \Big\} \; 5$$

NAME________________

(20) 9. Let S be the parametrized surface

$$x = u^2, \quad y = 2^{\frac{1}{2}}uv, \quad z = v^2 \quad \text{for } 0 \le u \le 1 \text{ and } 0 \le v \le 1 .$$

a) Calculate the area of S , $\vec{r} = u^2\vec{i} + \sqrt{2}uv\vec{j} + v^2\vec{k}$

$\vec{r}_u = 2u\vec{i} + \sqrt{2}v\vec{j}, \quad \vec{r}_v = \sqrt{2}u\vec{j} + 2v\vec{k}$

4 $\left\{ \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2u & \sqrt{2}v & 0 \\ 0 & \sqrt{2}u & 2v \end{vmatrix} = 2\sqrt{2}v^2\vec{i} - 4uv\vec{j} + 2\sqrt{2}u^2\vec{k} \right.$

4 $\boxed{|\vec{r}_u \times \vec{r}_v| = \sqrt{8v^4 + 16u^2v^2 + 8u^4} = \sqrt{8}\,(v^4 + 2u^2v^2 + u^4)^{\frac{1}{2}} = \sqrt{8}\,((u^2+v^2)^2)^{\frac{1}{2}} = \sqrt{8}\,(u^2+v^2)}$

4 $\left\{ \text{so Area} = \int_0^1\int_0^1 \sqrt{8}(u^2+v^2)\,du\,dv = \sqrt{8}\int_0^1 \frac{u^3}{3} + v^2u \Big|_{u=0}^{u=1} dv = \sqrt{8}\int_0^1 \left(\frac{1}{3} + v^2\right)dv = \sqrt{8}\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2\sqrt{8}}{3} \right.$

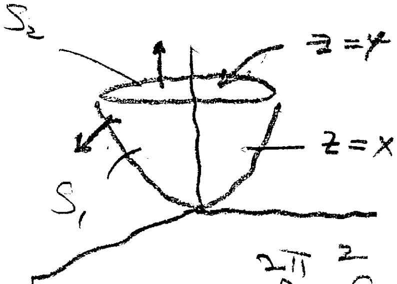
b) Evaluate $\iint_S y\, dS$

$$\left. = \int_0^1\int_0^1 \sqrt{2}uv\,\sqrt{8}\,(u^2+v^2)\,du\,dv \right\} 4$$

$$\left. \begin{aligned} &= 4\int_0^1\int_0^1 (u^3v + uv^3)\,du\,dv \\ &= 4\int_0^1 \frac{u^4}{4}v + \frac{u^2}{2}v^3 \Big|_{u=0}^{u=1} dv \\ &= 4\int_0^1 \left(\frac{1}{4}v + \frac{1}{2}v^3\right)dv = 4\left(\frac{1}{8} + \frac{1}{8}\right) = 1 \end{aligned} \right\} 4$$

NAME____________

(20) 10. Let E be the 3D-region which is enclosed by the surfaces $z = x^2 + y^2$ and $z = 4$ and let S be its boundary with outward pointing normal. If $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ verify the divergence theorem in this case by calculating both sides of the equation and seeing that they are equal.



$$\operatorname{div}\vec{F} = 1+1+1 = 3 \quad (2)$$

$$\iiint_E 3\,dV = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} 3\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 3(4-r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi}\int_0^{2}(4r-r^3)\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\Big|_0^2\,d\theta = 3\int_0^{2\pi}4\,d\theta = 24\pi \quad (5)$$

S_1: $z = x^2+y^2$, $0 \le z \le 4$, $\vec{n}$ downward pointing

$$\frac{\partial z}{\partial x} = 2x,\quad \frac{\partial z}{\partial y} = 2y$$

$$\iint_{S_2}\vec{F}\cdot\vec{n}\,dS = -\iint_D\left[(x)(-2x)+(y)(-2y)+z\right]dx\,dy$$

$$= \iint_D 2(x^2+y^2)-(x^2+y^2)\,dx\,dy = \int_0^{2\pi}\int_0^{2} r^2\cdot r\,dr\,d\theta = 8\pi \quad (6)$$

(5) S_2: $z=4$, $0 \le x^2+y^2 \le 4$, $\vec{n} = \vec{k}$

so $\vec{F}\cdot\vec{n} = z$ and $\iint_{S_2}\vec{F}\cdot\vec{n}\,dS = \iint_{S_2} z\,dS = \iint_{S_2} 4\,dS = 4 \text{ area } S_2 = (4)(4\pi) = 16\pi$

So

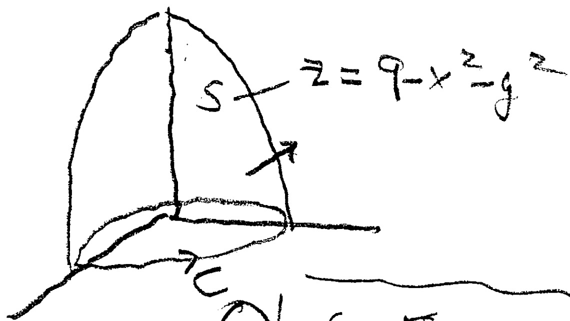
$$8\pi + 16\pi = 24\pi \quad (2)$$

NAME_______________

(20) 11. Let S be that part of the paraboloid $z = 9 - x^2 - y^2$ which is above the xy-plane with the upward pointing normal and C be its positively oriented boundary. Let

$$\mathbf{F} = (2z - y)\mathbf{i} + (x + z)\mathbf{j} + (3x - 2y)\mathbf{k}\,.$$

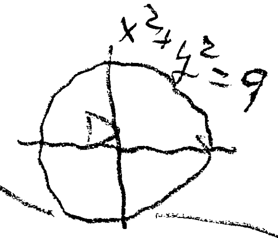
Verify Stokes' theorem in this case by calculating both sides of the equation and seeing that they are equal.



$$\operatorname{Curl} F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2z-y, & x+z, & 3x-2y \end{vmatrix} \qquad \langle -3, -1, 2 \rangle$$

(4) $\boxed{\operatorname{Curl} \vec{F} = (-2-1)\vec{i} - (3-2)\vec{j} + (1+1)\vec{k} = -3\vec{i} - \vec{j} + 2\vec{k}}$

(6) $S: z = 9 - x^2 - y^2, \quad \frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y$



$$\iint_S \operatorname{Curl} \vec{F} \cdot \vec{n}\, dS = (2) \iint_D [(-3)(2x) - (2y) + 2]\, dx\, dy$$

$$= \iint_D (-6x - 2y + 2)\, dx\, dy = \iint_D 2\, dx\, dy = 2 \text{ area } D = (2)(9\pi) = 18\pi$$

$C: \; x = 3\cos t, \; y = 3\sin t, \; z = 0, \; 0 \le t \le 2\pi$

$dx = -3\sin t\, dt, \; dy = 3\cos t\, dt, \; dz = 0$ (5)

$$\int_C (2\cancel{z}^{\,0} - y)\, dx + (x + \cancel{z}^{\,0})\, dy + (3x - 2y)\, \cancel{dz}_{\,0}$$

$$= \int_C -y\, dx + x\, dy = \int_0^{2\pi} (9\sin^2 t + 9\sin^2 t)\, dt$$

$$= \int_0^{2\pi} 9\, dt = 18\pi \quad (5)$$