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Dynamical systems method for solving linear finite-rank operator equations

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Abstract. A version of the dynamical systems method (DSM) for solving ill-conditioned linear algebraic systems is studied. An *a priori* and an *a posteriori* stopping rules are justified. An iterative scheme is constructed for solving ill-conditioned linear algebraic systems.

1. Introduction. We want to solve stably the equation

$$Au = f, \tag{1}$$

where A is a bounded linear operator in a real Hilbert space H. We assume that (1) has a solution, possibly nonunique, and denote by y the unique minimal-norm solution to (1), $y \perp \mathcal{N} := \mathcal{N}(A) := \{u : Au = 0\}$, $Ay = f$. We assume that the range of A, written $R(A)$, is not closed, so problem (1) is ill-posed. Let f_δ, $\|f - f_\delta\| \le \delta$, be the noisy data. We want to construct a stable approximation of y, given $\{\delta, f_\delta, A\}$. There are many methods for doing this: see, e.g., [9]–[12], [20], [21], to mention some (of the many) books, where variational regularization, quasisolutions, quasiinversion, and iterative regularization are studied, and [12]–[17], where the dynamical systems method (DSM) is studied systematically (see also [1], [20], [19], and references therein for related results). Recent papers on DSM are [18] and [4]–[8].

The basic new results of this paper are: 1) a new version of the DSM for solving equation (1) is justified; 2) a stable method for solving equation (1) with noisy data by the DSM is given; *a priori* and *a posteriori* stopping rules are proposed and justified; 3) an iterative method for solving linear ill-conditioned algebraic systems, based on the proposed version of DSM, is formulated; its convergence is proved; 4) numerical results are given; these results show that the proposed method yields a good alternative to some of

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the standard methods (e.g., to variational regularization, Landweber iterations, and some other methods).

The DSM version we study in this paper consists of solving the Cauchy problem

$$\dot{u}(t) = -P(Au(t) - f), \qquad u(0) = u_0, \qquad u_0 \perp \mathcal{N}, \qquad \dot{u} := \frac{du}{dt}, \tag{2}$$

and proving the existence of the limit $\lim_{t\to\infty} u(t) = u(\infty)$, and the relation $u(\infty) = y$, i.e.,

$$\lim_{t\to\infty} \|u(t) - y\| = 0. \tag{3}$$

Here P is a bounded operator such that $T := PA \geq 0$ is selfadjoint and $\mathcal{N}(T) = \mathcal{N}(A)$.

For any linear (not necessarily bounded) operator A there exists a bounded operator P such that $T = PA \geq 0$. For example, if $A = U|A|$ is the polar decomposition of A, then $|A| := (A^*A)^{1/2}$ is a selfadjoint operator, $T := |A| \geq 0$, U is a partial isometry, $\|U\| = 1$, and if $P := U^*$, then $\|P\| = 1$ and $PA = T$. Another choice of P, namely, $P = (A^*A + aI)^{-1}A^*$, $a = \text{const} > 0$, is used in Section 3. For this choice $Q := AP \geq 0$.

If the noisy data f_δ are given, $\|f_\delta - f\| \leq \delta$, then we solve the problem

$$\dot{u}_\delta(t) = -P(Au_\delta(t) - f_\delta), \qquad u_\delta(0) = u_0, \tag{4}$$

and prove that, for a suitable stopping time t_δ, and $u_\delta := u_\delta(t_\delta)$, one has

$$\lim_{\delta\to 0} \|u_\delta - y\| = 0. \tag{5}$$

An *a priori* and an *a posteriori* methods for choosing t_δ are given.

In Section 2 these results are formulated and recipes for choosing t_δ are proposed. In Section 3 a numerical example is presented.

2. Formulation of results. Suppose $A : H \to H$ is a bounded linear operator in a real Hilbert space H. Assume that equation (1) has a solution, not necessarily unique. Denote by y the unique minimal-norm solution, i.e., $y \perp \mathcal{N} := \mathcal{N}(A)$. Consider the DSM (2) where $u_0 \perp \mathcal{N}$ is arbitrary. Define

$$T := PA, \qquad Q := AP. \tag{6}$$

The unique solution to (2) is

$$u(t) = e^{-tT}u_0 + e^{-tT}\int_0^t e^{sT}\, ds\, Pf. \tag{7}$$

Let us first show that any ill-posed linear equation (1) with exact data can be solved by the DSM. We assume below that $P = (A^*A + aI)^{-1}A^*$, where $a = \text{const} > 0$. With this choice of P one has $\mathcal{N}(T) = \mathcal{N}(A)$ and $\|T\| \leq 1$.

2.1. *Exact data.* The following result is known (see [12]) but a short proof is included for completeness.

Theorem 1. *Suppose $u_0 \perp \mathcal{N}$ and $T^* = T \geq 0$. Then problem* (2) *has a unique solution defined on* $[0, \infty)$, *and* $u(\infty) = y$, *where* $u(\infty) = \lim_{t\to\infty} u(t)$.

Proof. Set $w := u(t) - y$ and $w_0 := w(0) = u_0 - y$. Note that $w_0 \perp \mathcal{N}$. One has

$$\dot{w} = -Tw, \qquad T := PA, \qquad w(0) = u_0 - y. \tag{8}$$

The unique solution to (8) is $w = e^{-tT} w_0$. Thus,

$$\|w\|^2 = \int_0^{\|T\|} e^{-2t\lambda}\, d\langle E_\lambda w_0, w_0\rangle,$$

where $\langle u, v\rangle$ is the inner product in H, and E_λ is the resolution of the identity of T. Thus,

$$\|w(\infty)\|^2 = \lim_{t\to\infty} \int_0^{\|T\|} e^{-2t\lambda}\, d\langle E_\lambda w_0, w_0\rangle = \|P_{\mathcal{N}} w_0\|^2 = 0,$$

where $P_{\mathcal{N}} = E_0 - E_{-0}$ is the orthogonal projector onto $\mathcal{N}$. Theorem 1 is proved. ■

2.2. *Noisy data f_δ.* Let us solve stably equation (1) assuming that f is not known, but f_δ, the noisy data, are known, where $\|f_\delta - f\| \leq \delta$. Consider the following DSM:

$$\dot{u}_\delta = -P(Au_\delta - f_\delta), \qquad u_\delta(0) = u_0. \tag{9}$$

Define

$$w_\delta := u_\delta - y, \qquad T := PA, \qquad w_\delta(0) = w_0 := u_0 - y \in \mathcal{N}^\perp.$$

We prove the following result:

Theorem 2. *If $T = T^* \geq 0$, $\lim_{\delta\to 0} t_\delta = \infty$, $\lim_{\delta\to 0} t_\delta \delta = 0$, and $w_0 \in \mathcal{N}^\perp$, then*

$$\lim_{\delta\to 0} \|w_\delta(t_\delta)\| = 0.$$

Proof. One has

$$\dot{w}_\delta = -Tw_\delta + \zeta_\delta, \qquad \zeta_\delta = P(f_\delta - f), \qquad \|\zeta_\delta\| \leq \|P\|\delta. \tag{10}$$

The unique solution of (10) is

$$w_\delta(t) = e^{-tT} w_\delta(0) + \int_0^t e^{-(t-s)T} \zeta_\delta\, ds.$$

Let us show that $\lim_{\delta\to 0}\|w_\delta(t_\delta)\|=0$. One has

$$\lim_{t\to\infty}\|w_\delta(t)\| \le \lim_{t\to\infty}\|e^{-tT}w_\delta(0)\| + \lim_{t\to\infty}\Big\|\int_0^t e^{-(t-s)T}\zeta_\delta\,ds\Big\|. \tag{11}$$

Let E_λ be the resolution of the identity corresponding to T. One uses the spectral theorem to get

$$\begin{aligned}\int_0^t e^{-(t-s)T}\,ds\,\zeta_\delta &= \int_0^t\int_0^{\|T\|} dE_\lambda\,\zeta_\delta e^{-(t-s)\lambda}\,ds = \int_0^{\|T\|} e^{-t\lambda}\frac{e^{t\lambda}-1}{\lambda}\,dE_\lambda\,\zeta_\delta\\ &= \int_0^{\|T\|}\frac{1-e^{-t\lambda}}{\lambda}\,dE_\lambda\,\zeta_\delta.\end{aligned} \tag{12}$$

Note that

$$0\le \frac{1-e^{-t\lambda}}{\lambda}\le t, \qquad \forall\lambda>0,\ t\ge 0, \tag{13}$$

since $1-x\le e^{-x}$ for $x\ge 0$. From (12) and (13), one obtains

$$\begin{aligned}\Big\|\int_0^t e^{-(t-s)T}\,ds\,\zeta_\delta\Big\|^2 &= \int_0^{\|T\|}\Big|\frac{1-e^{-t\lambda}}{\lambda}\Big|^2 d\langle E_\lambda\zeta_\delta,\zeta_\delta\rangle\\ &\le t^2\int_0^{\|T\|} d\langle E_\lambda\zeta_\delta,\zeta_\delta\rangle = t^2\|\zeta_\delta\|^2.\end{aligned} \tag{14}$$

This estimate also follows from the inequality $\|e^{-(t-s)T}\|\le 1$, which holds for $T^*=T\ge 0$ and $t\ge s$. Indeed, one has $\|\int_0^t e^{-(t-s)T}\,ds\|\le t$, and estimate (14) follows.

Since $\|\zeta_\delta\|\le\|P\|\delta$, from (11) and (14), one gets

$$\lim_{\delta\to 0}\|w_\delta(t_\delta)\| \le \lim_{\delta\to 0}\left(\|e^{-t_\delta T}w_\delta(0)\| + t_\delta\delta\|P\|\right) = 0.$$

Here we have used the relation

$$\lim_{\delta\to 0}\|e^{-t_\delta T}w_\delta(0)\| = \|P_{\mathcal{N}}w_0\| = 0,$$

where the last equality holds because $w_0\in\mathcal{N}^\perp$. Theorem 2 is proved. ■

From Theorem 2, it follows that the relation

$$t_\delta = \frac{C}{\delta^\gamma}, \qquad \gamma = \text{const}, \qquad \gamma\in(0,1),$$

where $C>0$ is a constant, can be used as an *a priori* stopping rule, i.e., for such t_δ one has

$$\lim_{\delta\to 0}\|u_\delta(t_\delta)-y\| = 0. \tag{15}$$

2.3. *Discrepancy principle.* In this section we assume that A is a linear finite-rank operator. Thus, it is a bounded linear operator. Let us consider equation (1) with noisy data f_δ, and a DSM of the form

$$\dot{u}_\delta = -PAu_\delta + Pf_\delta, \qquad u_\delta(0) = u_0, \tag{16}$$

for solving this equation. Equation (16) has been used in Section 2.2. Recall that y denotes the minimal-norm solution of (1), and that $\mathcal{N}(T) = \mathcal{N}(A)$ with our choice of P.

THEOREM 3. *Let $T := PA$ and $Q := AP$. Assume that $\|Au_0 - f_\delta\| > C\delta$ and $Q = Q^* \geq 0$, $T^* = T \geq 0$, and T is a finite-rank operator. Then the solution t_δ to the equation*

$$h(t) := \|Au_\delta(t) - f_\delta\| = C\delta, \qquad C = \text{const}, \qquad C \in (1,2), \tag{17}$$

does exist, is unique, $\lim_{\delta\to 0} t_\delta = \infty$, *and*

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0, \tag{18}$$

where y is the unique minimal-norm solution to (1)*.*

Proof. Define

$$v_\delta(t) := Au_\delta(t) - f_\delta, \qquad w(t) := u(t) - y, \qquad w_0 := u_0 - y.$$

One has

$$\begin{aligned} \frac{d}{dt}\|v_\delta(t)\|^2 &= 2\langle A\dot{u}_\delta(t), Au_\delta(t) - f_\delta\rangle \\ &= 2\langle A[-P(Au_\delta(t) - f_\delta)], Au_\delta(t) - f_\delta\rangle \\ &= -2\langle AP(Au_\delta - f_\delta), Au_\delta - f_\delta\rangle \leq 0, \end{aligned} \tag{19}$$

where the last inequality holds because $AP = Q \geq 0$. Thus, $\|v_\delta(t)\|$ is a nonincreasing function.

Let us prove that equation (17) has a solution for $C \in (1,2)$. One has the following commutation formulas:

$$e^{-sT}P = Pe^{-sQ}, \qquad Ae^{-sT} = e^{-sQ}A.$$

Using these formulas and the representation

$$u_\delta(t) = e^{-tT}u_0 + \int_0^t e^{-(t-s)T}Pf_\delta\, ds,$$

one gets

$$\begin{aligned} v_\delta(t) = Au_\delta(t) - f_\delta &= Ae^{-tT}u_0 + A\int_0^t e^{-(t-s)T}Pf_\delta\,ds - f_\delta \\ &= e^{-tQ}Au_0 + e^{-tQ}\int_0^t e^{sQ}\,ds\,Qf_\delta - f_\delta \\ &= e^{-tQ}A(u_0 - y) + e^{-tQ}f + e^{-tQ}(e^{tQ} - I)f_\delta - f_\delta \\ &= e^{-tQ}Aw_0 - e^{-tQ}f_\delta + e^{-tQ}f = e^{-tQ}Au_0 - e^{-tQ}f_\delta. \end{aligned} \tag{20}$$

Note that

$$\lim_{t\to\infty} e^{-tQ}Aw_0 = \lim_{t\to\infty} Ae^{-tT}w_0 = AP_{\mathcal{N}}w_0 = 0.$$

Here the continuity of A and the relation

$$\lim_{t\to\infty} e^{-tT}w_0 = \lim_{t\to\infty}\int_0^{\|T\|} e^{-st}\,dE_s\,w_0 = (E_0 - E_{-0})w_0 = P_{\mathcal{N}}w_0$$

were used. Therefore,

$$\lim_{t\to\infty}\|v_\delta(t)\| = \lim_{t\to\infty}\|e^{-tQ}(f - f_\delta)\| \le \|f - f_\delta\| \le \delta, \tag{21}$$

where $\|e^{-tQ}\| \le 1$ because $Q \ge 0$. The function $h(t)$ is continuous on $[0,\infty)$, $h(0) = \|Au_0 - f_\delta\| > C\delta$ and $h(\infty) \le \delta$. Thus, equation (17) must have a solution t_δ.

Let us prove the uniqueness of t_δ. If t_δ is nonunique, then without loss of generality we can assume that there exists $t_1 > t_\delta$ such that $\|Au_\delta(t_1) - f_\delta\| = C\delta$. Since $\|v_\delta(t)\|$ is nonincreasing and $\|v_\delta(t_\delta)\| = \|v_\delta(t_1)\|$, one has

$$\|v_\delta(t)\| = \|v_\delta(t_\delta)\|, \quad \forall t \in [t_\delta, t_1].$$

Thus,

$$\frac{d}{dt}\|v_\delta(t)\|^2 = 0, \quad \forall t \in (t_\delta, t_1). \tag{22}$$

Using (19) and (22) one obtains

$$\|\sqrt{AP}\,(Au_\delta(t) - f_\delta)\|^2 = \langle AP(Au_\delta(t) - f_\delta), Au_\delta(t) - f_\delta\rangle = 0, \quad \forall t \in [t_\delta, t_1],$$

where $\sqrt{AP} = Q^{1/2} \ge 0$ is well defined since $Q = Q^* \ge 0$. This implies that $Q^{1/2}(Au_\delta - f_\delta) = 0$. Thus

$$Q(Au_\delta(t) - f_\delta) = 0, \quad \forall t \in [t_\delta, t_1]. \tag{23}$$

From (20) one gets

$$v_\delta(t) = Au_\delta(t) - f_\delta = e^{-tQ}Au_0 - e^{-tQ}f_\delta. \tag{24}$$

Since $Qe^{-tQ} = e^{-tQ}Q$ and e^{-tQ} is an isomorphism, equalities (23) and (24) imply

$$Q(Au_0 - f_\delta) = 0.$$

This and (24) imply

$$AP(Au_\delta(t) - f_\delta) = e^{-tQ}(QAu_0 - Qf_\delta) = 0, \qquad t \geq 0.$$

Hence (19) yields

$$\frac{d}{dt}\|v_\delta\|^2 = 0, \qquad t \geq 0. \tag{25}$$

Consequently,

$$C\delta < \|Au_\delta(0) - f_\delta\| = \|v_\delta(0)\| = \|v_\delta(t_\delta)\| = \|Au_\delta(t_\delta) - f_\delta\| = C\delta.$$

This is a contradiction which proves the uniqueness of t_δ.

Let us prove (18). First, we have the following estimate:

$$\begin{aligned} \|Au(t_\delta) - f\| &\leq \|Au(t_\delta) - Au_\delta(t_\delta)\| + \|Au_\delta(t_\delta) - f_\delta\| + \|f_\delta - f\| \\ &\leq \left\| e^{-t_\delta Q} \int_0^{t_\delta} e^{sQ} Q \, ds \right\| \|f_\delta - f\| + C\delta + \delta, \end{aligned} \tag{26}$$

where $u(t)$ solves (2) and $u_\delta(t)$ solves (9). One uses the inequality

$$\left\| e^{-t_\delta Q} \int_0^{t_\delta} e^{sQ} Q \, ds \right\| = \|I - e^{-t_\delta Q}\| \leq 2,$$

and concludes from (26) that

$$\lim_{\delta \to 0} \|Au(t_\delta) - f\| = 0. \tag{27}$$

Secondly, we claim that

$$\lim_{\delta \to 0} t_\delta = \infty.$$

Suppose the contrary. Then there exist $t_0 > 0$ and a sequence $(t_{\delta_n})_{n=1}^\infty$ with $t_{\delta_n} < t_0$ and $\lim_{n\to\infty} \delta_n = 0$ such that

$$\lim_{n \to \infty} \|Au(t_{\delta_n}) - f\| = 0. \tag{28}$$

Analogously to (19), one proves that

$$\frac{d}{dt}\|v\|^2 \leq 0,$$

where $v(t) := Au(t) - f$. Thus, $\|v(t)\|$ is nonincreasing. This and (28) imply the relation $\|v(t_0)\| = \|Au(t_0) - f\| = 0$. Thus,

$$0 = v(t_0) = e^{-t_0 Q} A(u_0 - y).$$

Therefore $A(u_0 - y) = e^{t_0 Q} e^{-t_0 Q} A(u_0 - y) = 0$, so $u_0 - y \in \mathcal{N}$. Since $u_0 - y \in \mathcal{N}^\perp$, it follows that $u_0 = y$. This is a contradiction because

$$C\delta \le \|Au_0 - f_\delta\| = \|f - f_\delta\| \le \delta, \qquad 1 < C < 2.$$

Thus,

$$\lim_{\delta \to 0} t_\delta = \infty. \tag{29}$$

To continue the proof of (18), notice that, from (20) and the relation $\|Au_\delta(t_\delta) - f_\delta\| = C\delta$, one has

$$\begin{aligned} C\delta t_\delta &= \|t_\delta e^{-t_\delta Q} A w_0 - t_\delta e^{-t_\delta Q}(f_\delta - f)\| \\ &\le \|t_\delta e^{-t_\delta Q} A w_0\| + \|t_\delta e^{-t_\delta Q}(f_\delta - f)\| \le \|t_\delta e^{-t_\delta Q} A w_0\| + t_\delta \delta. \end{aligned} \tag{30}$$

We claim that

$$\lim_{\delta \to 0} t_\delta e^{-t_\delta Q} A w_0 = \lim_{\delta \to 0} t_\delta A e^{-t_\delta T} w_0 = 0. \tag{31}$$

Observe that (31) holds if $T \ge 0$ *has finite rank, and* $w_0 \in \mathcal{N}^\perp$. It also holds if $T \ge 0$ is compact and the Fourier coefficients $w_{0j} := \langle w_0, \phi_j \rangle$, $T\phi_j = \lambda_j \phi_j$, decay sufficiently fast. In this case

$$\begin{aligned} \|Ae^{-tT} w_0\|^2 &\le \|T^{1/2} e^{-tT} w_0\|^2 \\ &= \sum_{j=1}^{\infty} \lambda_j e^{-2\lambda_j t} |w_{0j}|^2 =: S = o(1/t^2), \qquad t \to \infty, \end{aligned}$$

provided that $\sum_{j=1}^{\infty} |w_{0j}| \lambda_j^{-2} < \infty$. Indeed,

$$S = \sum_{\lambda_j \le 1/t^{2/3}} + \sum_{\lambda_j > 1/t^{2/3}} =: S_1 + S_2.$$

One has

$$S_1 \le \frac{1}{t^2} \sum_{\lambda_j \le t^{-2/3}} \frac{|w_{0j}|^2}{\lambda_j^2} = o(1/t^2), \quad S_2 \le c e^{-2t^{1/3}} = o\left(\frac{1}{t^2}\right), \qquad t \to \infty,$$

where $c > 0$ is a constant.

From (31) and (30), one gets

$$0 \le \lim_{\delta \to 0} (C-1)\delta t_\delta \le \lim_{\delta \to 0} \|t_\delta e^{-t_\delta Q} A w_0\| = 0.$$

Thus,

$$\lim_{\delta \to 0} \delta t_\delta = 0. \tag{32}$$

Now, the desired conclusion (18) follows from (29), (32) and Theorem 2. Theorem 3 is proved. ■

2.4. *An iterative scheme.* Let us solve stably equation (1) assuming that f is not known, but f_δ, the noisy data, are known, where $\|f_\delta - f\| \le \delta$. Consider the following discrete version of the DSM:

$$u_{n+1,\delta} = u_{n,\delta} - hP(Au_{n,\delta} - f_\delta), \qquad u_{\delta,0} = u_0. \tag{33}$$

Define $u_n := u_{n,\delta}$ when $\delta \ne 0$, and set

$$w_n := u_n - y, \qquad T := PA, \qquad w_0 := u_0 - y \in \mathcal{N}^\perp.$$

Let $n = n_\delta$ be the stopping rule for iterations (33). Let us prove the following result:

Theorem 4. *Assume that* $T = T^* \ge 0$, $h\|T\| < 2$, $\lim_{\delta\to 0} n_\delta h = \infty$, $\lim_{\delta\to 0} n_\delta h\delta = 0$, *and* $w_0 \in \mathcal{N}^\perp$. *Then*

$$\lim_{\delta\to 0} \|w_{n_\delta}\| = \lim_{\delta\to 0} \|u_{n_\delta} - y\| = 0. \tag{34}$$

Proof. One has

$$\begin{aligned} &w_{n+1} = w_n - hTw_n + h\zeta_\delta, \qquad w_0 = u_0 - y,\\ &\zeta_\delta = P(f_\delta - f), \qquad \|\zeta_\delta\| \le \|P\|\delta. \end{aligned} \tag{35}$$

The unique solution of (35) is

$$w_{n+1} = (I - hT)^{n+1} w_0 + h\sum_{i=0}^{n} (I - hT)^i \zeta_\delta.$$

We show that $\lim_{\delta\to 0} \|w_{n_\delta}\| = 0$. One has

$$\|w_n\| \le \|(I - hT)^n w_0\| + \Big\| h\sum_{i=0}^{n-1} (I - hT)^i \zeta_\delta \Big\|. \tag{36}$$

Let E_λ be the resolution of the identity corresponding to T. One uses the spectral theorem to get

$$\begin{aligned} h\sum_{i=0}^{n-1} (I - hT)^i &= h\sum_{i=0}^{n-1} \int_0^{\|T\|} (1 - h\lambda)^i \, dE_\lambda = h\int_0^{\|T\|} \frac{1 - (1 - \lambda h)^n}{1 - (1 - h\lambda)} \, dE_\lambda \\ &= \int_0^{\|T\|} \frac{1 - (1 - \lambda h)^n}{\lambda} \, dE_\lambda. \end{aligned} \tag{37}$$

Note that

$$0 \le \frac{1 - (1 - h\lambda)^n}{\lambda} \le hn, \qquad \forall \lambda > 0,\ t \ge 0, \tag{38}$$

since $1-(1-\alpha)^n \le \alpha n$ for all $\alpha \in [0,2]$. From (37) and (38), one obtains

$$\Big\| h\sum_{i=0}^{n-1}(I-hT)^i\zeta_\delta\Big\|^2 = \int\limits_0^{\|T\|}\Big|\frac{1-(1-\lambda h)^n}{\lambda}\Big|^2 d\langle E_\lambda\zeta_\delta,\zeta_\delta\rangle \le (hn)^2\int\limits_0^{\|T\|} d\langle E_\lambda\zeta_\delta,\zeta_\delta\rangle = (nh)^2\|\zeta_\delta\|^2. \tag{39}$$

Alternatively, this estimate follows from the inequality $\|(I-hT)^i\| \le 1$, provided that $0 \le hT < 2$. Indeed, in this case $\|\sum_{i=0}^{n-1}(I-hT)^i\| \le n$, and this implies (39).

Since $\|\zeta_\delta\| \le \|P\|\delta$, from (36) and (39), one gets

$$\lim_{\delta\to 0}\|w_{n_\delta}\| \le \lim_{\delta\to 0}\left(\|(I-hT)^{n_\delta}w_\delta(0)\| + hn_\delta\delta\|P\|\right) = 0.$$

Here we have used the relation

$$\lim_{\delta\to 0}\|(I-hT)^{n_\delta}w_\delta(0)\| = \|P_{\mathcal{N}}w_0\| = 0,$$

and the last equality holds because $w_0 \in \mathcal{N}^\perp$. Theorem 4 is proved. ■

From Theorem 4, it follows that the relation

$$n_\delta = \frac{C}{h\delta^\gamma}, \qquad \gamma = \text{const},\ \gamma\in(0,1),$$

where $C>0$ is a constant, can be used as an *a priori* stopping rule, i.e., for such n_δ one has

$$\lim_{\delta\to 0}\|u_{n_\delta} - y\| = 0. \tag{40}$$

2.5. *An iterative scheme with a stopping rule based on a discrepancy principle.* In this section we assume that A is a finite-rank linear operator. Thus, it is a bounded linear operator. Let us consider equation (1) with noisy data f_δ, and a DSM of the form

$$u_{n+1} = u_n - hP(Au_n - f_\delta), \qquad u_n|_{n=0} = u_0, \tag{41}$$

for solving this equation. Here u_0 is an arbitrary initial approximation. Equation (41) has been used in Section 2.4. Recall that y denotes the minimal-norm solution of equation (1). An example of a choice of P is given in Section 3.

Note that $\mathcal{N} := \mathcal{N}(T) = \mathcal{N}(A)$.

Theorem 5. *Let $T := PA$ and $Q := AP$. Assume that $\|Au_0 - f_\delta\| > C\delta$, $Q = Q^* \ge 0$, $T^* = T \ge 0$, $h\|T\| < 2$, $h\|Q\| < 2$, and T is a finite-rank operator. Then there exists a unique n_δ such that*

$$\|Au_{n_\delta} - f_\delta\| \le C\delta < \|Au_{n_\delta - 1} - f_\delta\|, \qquad C = \text{const},\ C\in(1,2). \tag{42}$$

For this n_δ one has

$$\lim_{\delta\to 0}\|u_{n_\delta}-y\|=0. \tag{43}$$

Proof. Define

$$v_n := Au_n - f_\delta, \quad w_n := u_n - y, \quad w_0 := u_0 - y.$$

From (41), one gets

$$v_{n+1} = Au_{n+1} - f_\delta = Au_n - f_\delta - hAP(Au_n - f_\delta) = v_n - hQv_n.$$

This implies

$$\begin{aligned}\|v_{n+1}\|^2 - \|v_n\|^2 &= \langle v_{n+1} - v_n, v_{n+1} + v_n\rangle \\ &= \langle -hQv_n, v_n - hQv_n + v_n\rangle \\ &= -\langle v_n, hQ(2-hQ)v_n\rangle \le 0\end{aligned} \tag{44}$$

where the last inequality holds because $AP = Q \ge 0$ and $\|hQ\| < 2$. Thus, $(\|v_n\|)_{n=1}^\infty$ is a nonincreasing sequence.

Let us prove that equation (42) has a solution for $C \in (1,2)$. One has the following commutation formulas:

$$(I-hT)^nP = P(I-hQ)^n, \quad A(I-hT)^n = (I-hQ)^nA.$$

Using these formulas, the representation

$$u_n = (I-hT)^n u_0 + h\sum_{i=0}^{n-1}(I-hT)^iPf_\delta,$$

and the identity $(I-B)\sum_{i=0}^{n-1}B^i = I - B^n$, with $B = I - hQ$, $I - B = hQ$, one gets

$$\begin{aligned}v_n = Au_n - f_\delta &= A(I-hT)^nu_0 + Ah\sum_{i=0}^{n-1}(I-hT)^iPf_\delta - f_\delta \\ &= (I-hQ)^nAu_0 + \sum_{i=0}^{n-1}(I-hQ)^ihQf_\delta - f_\delta \\ &= (I-hQ)^nAu_0 - (I-(I-hQ)^n)f_\delta - f_\delta \\ &= (I-hQ)^n(Au_0 - f) + (I-hQ)^n(f-f_\delta) \\ &= (I-hQ)^nAw_0 + (I-hQ)^n(f-f_\delta).\end{aligned} \tag{45}$$

Let $V := hQ$. If $V = V^* \ge 0$ is an operator with $\|V\| \le 2$, then $\|I - V\| = \sup_{0\le s\le 2}|1-s| \le 1$. Thus, $\|I - hQ\| \le 1$.

Note that

$$\lim_{n\to\infty}(I-hQ)^nAw_0 = \lim_{n\to\infty}A(I-hT)^nw_0 = AP_{\mathcal{N}}w_0 = 0,$$

where $P_{\mathcal{N}}$ is the orthoprojection onto the null-space $\mathcal{N}$ of the operator T, and where the continuity of A and the relation

$$\lim_{n\to\infty}(I-hT)^n w_0=\lim_{n\to\infty}\int\limits_0^{\|T\|}(1-sh)^n\,dE_s\,w_0=(E_0-E_{-0})w_0=P_{\mathcal{N}}w_0$$

for $0\le sh<2$ were used. Therefore,

$$\lim_{n\to\infty}\|v_\delta(t)\|=\lim_{n\to\infty}\|(I-hQ)^n(f-f_\delta)\|\le\|f-f_\delta\|\le\delta, \tag{46}$$

where $\|I-hQ\|\le 1$ because $Q\ge 0$ and $\|hQ\|<2$. The sequence $\{\|v_n\|\}_{n=1}^\infty$ is nonincreasing with $\|v_0\|>C\delta$ and $\lim_{n\to\infty}\|v_n\|\le\delta$. Thus, there exists $n_\delta>0$ such that (42) holds.

Let us prove (43). Let $u_{n,0}$ be the sequence defined by the relations

$$u_{n+1,0}=u_{n,0}-hP(Au_{n,0}-f),\qquad u_{0,0}=u_0.$$

First, we have the following estimate:

$$\begin{aligned}\|Au_{n_\delta,0}-f\|&\le\|Au_{n_\delta}-Au_{n_\delta,0}\|+\|Au_{n_\delta}-f_\delta\|+\|f_\delta-f\|\\&\le\Big\|\sum_{i=0}^{n_\delta-1}(I-hQ)^i hQ\Big\|\|f_\delta-f\|+C\delta+\delta.\end{aligned} \tag{47}$$

Since $0\le hQ<2$, one has $\|I-hQ\|\le 1$. This implies

$$\Big\|\sum_{i=0}^{n_\delta-1}(I-hQ)^i hQ\Big\|=\|I-(I-hQ)^{n_\delta}\|\le 2,$$

and one concludes from (47) that

$$\lim_{\delta\to0}\|Au_{n_\delta,0}-f\|=0. \tag{48}$$

Secondly, we claim that

$$\lim_{\delta\to0}hn_\delta=\infty.$$

Suppose the contrary. Then there exist $n_0>0$ and a sequence $(n_{\delta_n})_{n=1}^\infty$ with $n_{\delta_n}<n_0$ such that

$$\lim_{n\to\infty}\|Au_{n_\delta,0}-f\|=0. \tag{49}$$

Analogously to (44), one proves that

$$\|v_{n,0}\|\le\|v_{n-1,0}\|,$$

where $v_{n,0}=Au_{n,0}-f$. Thus, the sequence $\|v_{n,0}\|$ is nonincreasing. This and (49) imply the relation $\|v_{n_0,0}\|=\|Au_{n_0,0}-f\|=0$. Thus,

$$0=v_{n_0,0}=(I-hQ)^{n_0}A(u_0-y).$$

This implies $A(u_0-y)=(I-hQ)^{-n_0}(I-hQ)^{n_0}A(u_0-y)=0$, so $u_0-y\in\mathcal{N}$. Since, by the assumption, $u_0-y\in\mathcal{N}^\perp$, it follows that $u_0=y$. This is a contradiction because

$$C\delta \le \|Au_0 - f_\delta\| = \|f - f_\delta\| \le \delta, \qquad 1 < C < 2.$$

Thus,

$$\lim_{\delta\to 0} hn_\delta = \infty. \tag{50}$$

Let us continue the proof of (43). From (45) and $\|Au_{n_\delta} - f_\delta\| = C\delta$, one has

$$\begin{aligned} C\delta n_\delta h &= \|n_\delta h(I-hQ)^{n_\delta}Aw_0 - n_\delta h(I-hQ)^{n_\delta}(f_\delta - f)\| \\ &\le \|n_\delta h(I-hQ)^{n_\delta}Aw_0\| + \|n_\delta h(I-hQ)^{n_\delta}(f_\delta - f)\| \\ &\le \|n_\delta h(I-hQ)^{n_\delta}Aw_0\| + n_\delta h\delta. \end{aligned} \tag{51}$$

We note that if $w_0 \in \mathcal{N}^\perp$, $0 \le hT < 2$, and T is a finite-rank operator, then

$$\lim_{\delta\to 0} n_\delta h(I-hQ)^{n_\delta}Aw_0 = \lim_{\delta\to 0} n_\delta hA(I-hT)^{n_\delta}w_0 = 0. \tag{52}$$

From (51) and (52) one gets

$$0 \le \lim_{\delta\to 0}(C-1)\delta hn_\delta \le \lim_{\delta\to 0}\|n_\delta h(I-hQ)^{n_\delta}Aw_0\| = 0.$$

Thus,

$$\lim_{\delta\to 0}\delta n_\delta h = 0. \tag{53}$$

Now (43) follows from (50), (53) and Theorem 4. Theorem 5 is proved. ■

3. Numerical experiments

3.1. *Computing $u_\delta(t_\delta)$.* In [3] the DSM (9) was investigated with $P = A^*$ and the singular value decomposition (SVD) of A was assumed known. In general, it is computationally expensive to get the SVD of large scale matrices. In this paper, we have derived an iterative scheme for solving ill-conditioned linear algebraic systems $Au = f_\delta$ without using SVD of A.

Choose $P = (A^*A + a)^{-1}A^*$ where a is a fixed positive constant. This choice of P satisfies all the conditions in Theorem 3. In particular, $Q = AP = A(A^*A + aI)^{-1}A^* = AA^*(AA^* + aI)^{-1} \ge 0$ is a selfadjoint operator, and $T = PA = (A^*A + aI)^{-1}A^*A \ge 0$ is a selfadjoint operator. Since

$$\|T\| = \left\| \int_0^{\|A^*A\|} \frac{\lambda}{\lambda + a}\, dE_\lambda \right\| = \sup_{0\le\lambda\le\|A^*A\|} \frac{\lambda}{\lambda + a} < 1,$$

where E_λ is the resolution of the identity of A^*A, the condition $h\|T\| < 2$ in Theorem 5 is satisfied for all $0 < h \le 1$. Set $h = 1$ and $P = (A^*A + a)^{-1}A^*$ in (41). Then one gets the following iterative scheme:

$$u_{n+1} = u_n - (A^*A + aI)^{-1}(A^*Au_n - A^*f_\delta), \qquad u_0 = 0. \tag{54}$$

We have chosen $u_0 = 0$ for simplicity. However, one may choose $u_0 = v_0$ if v_0 is known to be a better approximation to y than 0 and $v_0 \in \mathcal{N}^{\perp}$. In iterations (54) we use a stopping rule of discrepancy type. Indeed, we stop the iterations if u_n satisfies the condition

$$\|Au_n - f_\delta\| \le 1.01\delta. \tag{55}$$

The choice of a affects both the accuracy and the computation time of the method. If a is too large, one needs more iterations to approach the desired accuracy, so the computation time will be large. If a is too small, then the results become less accurate because for a too small the inversion of the operator $A^*A + aI$ is an ill-posed problem since the operator A^*A is not boundedly invertible. Using the idea of the choice of the initial guess of the regularization parameter from [2], we choose a to satisfy the condition

$$\delta \le \phi(a) := \|A(A^*A + a)^{-1}A^*f_\delta - f_\delta\| \le 2\delta. \tag{56}$$

This can be done by using the following strategy:

1. Choose $a := \delta\|A\|^2/(3\|f_\delta\|)$ as an initial guess for a.
2. Compute $\phi(a)$. If a satisfies (56), then we are done. Otherwise, go to Step 3.
3. If $c = \phi(a)/\delta > 3$, replace a by $a/2(c-1)$ and go back to Step 2. If $2 < c \le 3$, then replace a by $a/2(c-1)$ and go back to Step 2. Otherwise, go to Step 4.
4. If $c = \phi(a)/\delta < 1$, then replace a by $3a$. If the inequality $c < 1$ has occurred in an earlier iteration, stop the iterations and use $3a$ as a in iterations (54). Otherwise, go back to Step 2.

In our experiments, we denote by DSM the iterative scheme (54), by VR$_i$ a Variational Regularization method (VR) with a as the regularization parameter, and by VR$_n$ the VR in which Newton's method is used for finding the regularization parameter from a discrepancy principle. We compare these methods in terms of relative error and number of iterations, denoted by n_{iter}.

All the experiments were carried out in the double arithmetics precision environment using MATLAB.

3.2. *A linear algebraic system related to an inverse problem for the heat equation.* In this section, we apply the DSM and the VR to solve a linear algebraic system used in [2]. This linear algebraic system is a part of numerical solution of an inverse problem for the heat equation. This problem reduces to a Volterra integral equation of the first kind with $[0,1]$ as the integration interval. The kernel is $K(s,t) = k(s-t)$ with

$$k(t) = \frac{t^{-3/2}}{2\kappa\sqrt{\pi}}\exp\left(-\frac{1}{4\kappa^2 t}\right).$$

Here, we use the value $\kappa = 1$. In [2] the integral equation was discretized by means of simple collocation and the midpoint rule with n points. The unique exact solution u_n was constructed, and then the right-hand side b_n was produced as $b_n = A_n u_n$ (see [2]). In our test, we use $n = 10, 20, \ldots, 100$ and $b_{n,\delta} = b_n + e_n$, where e_n is a vector containing random entries, normally distributed with mean 0, variance 1, and scaled so that $\|e_n\| = \delta_{\text{rel}}\|b_n\|$. This linear system is ill-posed: the condition number of A_{100} obtained by using the function *cond* provided by MATLAB is $1.3717 \cdot 10^{37}$. This shows that the corresponding linear algebraic system is severely ill-conditioned.

Table 1. Numerical results for the inverse heat equation with $\delta_{\text{rel}} = 0.05$, $n = 10i$, $i = \overline{1,10}$.

	DSM		VR$_i$		VR$_n$	
n	n_{iter}	$\|u_\delta - y\|_2/\|y\|_2$	n_{iter}	$\|u_\delta - y\|_2/\|y\|_2$	n_{iter}	$\|u_\delta - y\|_2/\|y\|_2$
10	3	0.1971	1	0.2627	5	0.2117
20	4	0.3359	1	0.4589	5	0.3551
30	4	0.3729	1	0.4969	5	0.3843
40	4	0.3856	1	0.5071	5	0.3864
50	5	0.3158	1	0.4789	6	0.3141
60	6	0.2892	1	0.4909	6	0.3060
70	7	0.2262	1	0.4792	8	0.2156
80	6	0.2623	1	0.4809	7	0.2600
90	5	0.2856	1	0.4816	7	0.2715
100	7	0.2358	1	0.4826	7	0.3405

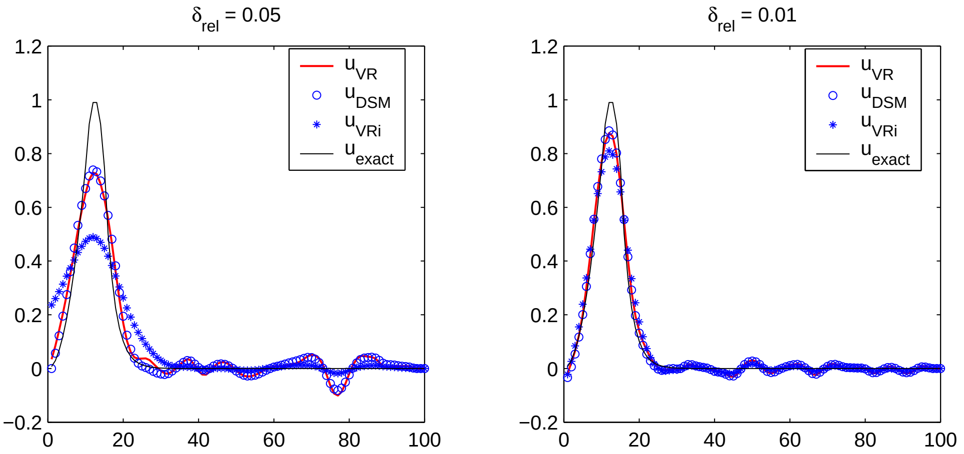

Fig. 1. Plots of solutions obtained by DSM and VR for the inverse heat equation when $n = 100$, $\delta_{\text{rel}} = 0.05$ (left) and $\delta_{\text{rel}} = 0.01$ (right)

Table 1 shows that the results obtained by the DSM are comparable to those by the VR_n in terms of accuracy. The time of computation of the DSM is also comparable to that of the VR_n. In some situations, the results by VR_n and the DSM are the same although the VR_n uses three more iterations than does the DSM. The conclusion from this table is that DSM competes favorably with the VR_n in both accuracy and time of computation.

Figure 1 plots numerical solutions to the inverse heat equation for $\delta_{\mathrm{rel}} = 0.05$ and $\delta_{\mathrm{rel}} = 0.01$ when $n = 100$. From the figure one can see that the numerical solutions obtained by the DSM are about the same as those by the VR_n. In these examples, the time of computation of the DSM is about the same as that of the VR_n.

The conclusion is that the DSM competes favorably with the VR_n in this experiment.

4. Concluding remarks. The iterative scheme (54) can be considered as a modification of the Landweber iterations. The difference between the two methods is in multiplication by $P = (A^*A+aI)^{-1}$. Our iterative method is much faster than the conventional Landweber iterations. The iterative method (54) is an analog of the Gauss–Newton method. It can be considered as a regularized Gauss–Newton method for solving ill-conditioned linear algebraic systems. The advantage of using (54) instead of using (4.1.3) in [2] is that one only has to compute the lower upper (LU) decomposition of A^*A+aI once while the algorithm in [2] requires computing LU at every step. Note that computing the LU is the main cost for solving a linear system. Numerical experiments show that the new method competes favorably with the VR in our experiments.

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