# Low-wavenumber forcing and turbulent energy dissipation

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## 1 Introduction

In many Direct Numerical Simulations (DNS) of turbulence researchers inject power into the fluid at large scales and then observe how it "propagates" to the small scales [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One such type of stirring is to take the force $\mathbf{f}(\mathbf{x}, t)$ to be proportional to the projection of the velocity $\mathbf{u}(\mathbf{x}, t)$ of the flow onto its lowest Fourier modes, while keeping the rate of injected external power constant. In this paper we perform a simple but rigorous analysis to establish bounds on the relationship between the energy dissipation rate (which is the same as the injected power) and the resulting Reynolds number. While this analysis cannot give detailed information of the energy spectrum, it does provide some indication of the balance of energy between the lower, directly forced, modes, and those excited by the cascade. This work is an extension of the analysis in [13, 14, 15], where the force is fixed (not a functional of the velocity).

Consider fluid in a periodic $d$-dimensional box of side length $\ell$. The allowed wave vectors $\mathbf{k}$ are of the form $\mathbf{k} = \frac{2\pi}{\ell}\mathbf{a}$, where $\mathbf{a} \in \mathbb{Z}^d$ is a $d$-dimensional vector with integer components. Let $\mathcal{L}$ be the subset of wave vectors that have the smallest possible wavenumber (namely, $\frac{2\pi}{\ell}$); $\mathcal{L}$ consists of $2d$ elements: $\mathcal{L} = \{\pm\frac{2\pi}{\ell}\mathbf{e}_1, \ldots, \pm\frac{2\pi}{\ell}\mathbf{e}_d\}$. The operator $\mathcal{P}$ projects the vector field

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}, t)\, \mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}}$$

onto the subspace spanned by the Fourier components with wave vectors in $\mathcal{L}$:

$$\mathcal{P}\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}\in\mathcal{L}} \hat{\mathbf{u}}(\mathbf{k}, t)\, \mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}} \ . \tag{1}$$

Obviously, $\mathcal{P}$ maps $L^2$ into $L^2$ vector fields; in fact, $\mathcal{P}\mathbf{u}$ is $C^\infty$ in the spatial variables. The projection also preserves the incompressibility property. That is, if $\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0$, then $\nabla \cdot \mathcal{P}\mathbf{u}(\mathbf{x}, t) = 0$.

The Navier-Stokes equation is

$$\dot{\mathbf{u}} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \nu\Delta\mathbf{u} + \mathbf{f} \ , \tag{2}$$

with $\mathbf{f}(\mathbf{x},t)$ taken in the form

$$\mathbf{f}(\mathbf{x},t) = \epsilon\,\frac{\mathcal{P}\mathbf{u}(\mathbf{x},t)}{\frac{1}{\ell^d}\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2^2} \ . \tag{3}$$

where $\|\cdot\|_2$ stands for the $L^2$-norm, $\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2 := \left[\int|\mathcal{P}\mathbf{u}(\mathbf{x},t)|^2\,\mathrm{d}^d\mathbf{x}\right]^{\frac{1}{2}}$.

This choice of forcing ensures that the input power is constant:

$$\int \mathbf{u}(\mathbf{x},t)\cdot\mathbf{f}(\mathbf{x},t)\,\mathrm{d}^d\mathbf{x} = \ell^d\epsilon \ . \tag{4}$$

In this approach $\epsilon$, $\nu$ and $\ell$ are the (only) control parameters. On average, the power input is the viscous energy dissipation rate:

$$\epsilon := \frac{1}{\ell^d}\int \mathbf{u}(\mathbf{x},t)\cdot\mathbf{f}(\mathbf{x},t)\,\mathrm{d}^d\mathbf{x} = \nu\,\frac{\langle\|\nabla\mathbf{u}\|_2^2\rangle}{\ell^d} \ , \tag{5}$$

where $\langle\cdot\rangle$ stands for the long time average. The non-dimensional measure of energy dissipation is defined as

$$\beta := \frac{\epsilon\ell}{U^3} \ , \tag{6}$$

which is a function of $\mathsf{Re} := \frac{U\ell}{\nu}$, the Reynolds number, where $U$ is the r.m.s. velocity defined by $U^2 := \frac{\langle\|\mathbf{u}\|_2^2\rangle}{\ell^d}$, a measure of the total kinetic energy of the fluid. Our analysis will establish limits on the relationship between $\beta$ and $\mathsf{Re}$.

Because we will study the "low-$k$" Fourier modes (i.e., modes with wave vectors in $\mathcal{L}$), we also introduce the r.m.s. velocity $V$ contained in these modes,

$$V^2 := \frac{\langle\|\mathcal{P}\mathbf{u}\|_2^2\rangle}{\ell^d} \ . \tag{7}$$

The bounds on the dissipation $\beta$ will be in terms of $\mathsf{Re}$ and the quantity

$$\mathfrak{p} := \frac{V}{U} \sim \sqrt{\frac{\text{“low-}k\text{” kinetic energy of the fluid}}{\text{Total kinetic energy of the fluid}}} \ . \tag{8}$$

The case $\mathfrak{p} \approx 1$ corresponds to laminar flow, when the turbulent cascade is inoperative.

## 2 Derivation of the bounds

### 2.1 Lower bounds on the energy dissipation

To obtain lower bounds on the energy dissipation, we proceed as usual by multiplying the Navier-Stokes equation (2) by $\mathbf{u}(\mathbf{x},t)$ and integrating over the volume of the fluid to obtain the instantaneous power balance,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{u}(\cdot,t)\right\|_2^2 = -\nu\left\|\nabla\mathbf{u}(\cdot,t)\right\|_2^2 + \ell^d\epsilon \ , \tag{9}$$

where $\left\|\nabla\mathbf{u}(\cdot,t)\right\|_2^2 := \int\left|\sum_{j,m=1}^d \partial_j u_m(\mathbf{x},t)\right|^2 \mathrm{d}^d\mathbf{x}$.

Now we use the facts that the lengths of wavevectors $\mathbf{k}\notin\mathcal{L}$ are at least $2\pi\sqrt{2}/\ell$, and that $\left\|\mathbf{u}(\cdot,t)-\mathcal{P}\mathbf{u}(\cdot,t)\right\|_2^2 = \left\|\mathbf{u}(\cdot,t)\right\|_2^2 - \left\|\mathcal{P}\mathbf{u}(\cdot,t)\right\|_2^2$, to derive a lower bound on $\left\|\nabla\mathbf{u}(\cdot,t)\right\|_2^2$:

$$\begin{aligned}
\left\|\nabla\mathbf{u}(\cdot,t)\right\|_2^2 &= \int\left|\nabla\mathbf{u}(\mathbf{x},t)\right|^2\mathrm{d}^d\mathbf{x} = \ell^d\sum_{\mathbf{k}} k^2|\hat{\mathbf{u}}(\mathbf{k},t)|^2 \\
&= \ell^d\left(\sum_{\mathbf{k}\in\mathcal{L}} k^2|\hat{\mathbf{u}}(\mathbf{k},t)|^2 + \sum_{\mathbf{k}\notin\mathcal{L}} k^2|\hat{\mathbf{u}}(\mathbf{k},t)|^2\right) \\
&\geq \ell^d\,\frac{4\pi^2}{\ell^2}\left(\sum_{\mathbf{k}\in\mathcal{L}}|\hat{\mathbf{u}}(\mathbf{k},t)|^2 + 2\sum_{\mathbf{k}\notin\mathcal{L}}|\hat{\mathbf{u}}(\mathbf{k},t)|^2\right) \\
&= \frac{4\pi^2}{\ell^2}\left(\left\|\mathcal{P}\mathbf{u}(\cdot,t)\right\|_2^2 + 2\left\|\mathbf{u}(\cdot,t)-\mathcal{P}\mathbf{u}(\cdot,t)\right\|_2^2\right) \\
&= \frac{4\pi^2}{\ell^2}\left(2\left\|\mathbf{u}(\cdot,t)\right\|_2^2 - \left\|\mathcal{P}\mathbf{u}(\cdot,t)\right\|_2^2\right) \ .
\end{aligned} \tag{10}$$

From (9) and (10) we obtain the differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{u}(\cdot,t)\right\|_2^2 \leq -\nu\,\frac{4\pi^2}{\ell^2}\left\|\mathbf{u}(\cdot,t)\right\|_2^2 + \ell^d\epsilon \ ,$$

from which, using Gronwall's inequality, we deduce

$$\frac{1}{2}\left\|\mathbf{u}(\cdot,t)\right\|_2^2 \leq \frac{1}{2}\left\|\mathbf{u}(\cdot,0)\right\|_2^2 \mathrm{e}^{-\frac{8\pi^2\nu}{\ell^2}t} + \ell^d\epsilon\frac{\ell^2}{8\pi^2\nu}\left(1-\mathrm{e}^{-\frac{8\pi^2\nu}{\ell^2}t}\right) \ . \tag{11}$$

The inequality (11) implies that $\left\|\mathbf{u}(\cdot,t)\right\|_2^2$ is bounded uniformly in time, which in turn implies that the time average of the time derivative in (9) vanishes. This ensures that the time-averaged power balance (assuming that the limit associated with the long time average exists) is indeed given by (5).

Taking the time average of (10), we obtain the bound

$$\frac{4\pi^2\nu}{\ell^2}\,(2U^2 - V^2) \le \epsilon\ ,$$

which in non-dimensional variables reads

$$\frac{4\pi^2}{\mathsf{Re}}(2 - \mathfrak{p}^2) \le \beta\ . \tag{12}$$

### 2.2 Upper bound on the energy dissipation

To derive an upper bound on $\beta$, we multiply the Navier-Stokes equation (2) by $\frac{\mathcal{P}\mathbf{u}(\mathbf{x},t)}{\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2}$ and integrate. The term with $\dot{\mathbf{u}}$ gives a total time derivative,

$$\int \dot{\mathbf{u}}\cdot\frac{\mathcal{P}\mathbf{u}}{\|\mathcal{P}\mathbf{u}\|_2}\,\mathrm{d}^d\mathbf{x} = \frac{1}{\|\mathcal{P}\mathbf{u}\|_2}\int\frac{\partial}{\partial t}(\mathcal{P}\mathbf{u})\cdot\mathcal{P}\mathbf{u}\,\mathrm{d}^d\mathbf{x} = \frac{1}{2}\frac{\mathrm{d}\,\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2}{\mathrm{d}t}\ .$$

For the viscosity term we obtain (integrating by parts)

$$\nu\int(\Delta\mathbf{u})\cdot\frac{\mathcal{P}\mathbf{u}}{\|\mathcal{P}\mathbf{u}\|_2}\,\mathrm{d}^d\mathbf{x} = -\nu\,\frac{\|\nabla\mathcal{P}\mathbf{u}\|_2^2}{\|\mathcal{P}\mathbf{u}\|_2} = -\nu\,\frac{4\pi^2}{\ell^2}\,\|\mathcal{P}\mathbf{u}\|_2\ ,$$

while the forcing term gives $\ell^d\epsilon/\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2$ (cf. (4)).

To estimate the inertial term, we introduce temporarily the notation $\mathbf{p}(\mathbf{x},t) := \mathcal{P}\mathbf{u}(\mathbf{x},t)$. We will make use of the uniform (in $\mathbf{x}$ and $t$) estimate

$$\frac{|\partial_j p_m(\mathbf{x},t)|}{\|\mathbf{p}(\cdot,t)\|_2} \le \sum_{\mathbf{k}\in\mathcal{L}}\frac{|k_j|\,|\hat{u}_m(\mathbf{k},t)|}{\|\mathbf{p}(\cdot,t)\|_2} \le \frac{2\pi}{\ell^{1+\frac{d}{2}}}\,\frac{\sum_{\mathbf{k}\in\mathcal{L}}|\hat{\mathbf{u}}(\mathbf{k},t)|}{\sqrt{\sum_{\mathbf{k}'\in\mathcal{L}}|\hat{\mathbf{u}}(\mathbf{k}',t)|^2}} \le \frac{2\pi\sqrt{d}}{\ell^{1+\frac{d}{2}}}\ . \tag{13}$$

Then the inertial term may be bounded (we use $\nabla\cdot\mathbf{p} = 0$):

$$\left|\int[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\frac{\mathbf{p}}{\|\mathbf{p}\|_2}\,\mathrm{d}^d\mathbf{x}\right| = \left|\int\mathbf{u}\cdot\frac{\nabla\mathbf{p}}{\|\mathbf{p}\|_2}\cdot\mathbf{u}\,\mathrm{d}^d\mathbf{x}\right|$$
$$\le \frac{2\pi\sqrt{d}}{\ell^{1+(d/2)}}\int|\mathbf{u}|^2\,\mathrm{d}^d\mathbf{x} = \frac{2\pi\sqrt{d}}{\ell^{1+(d/2)}}\,\|\mathbf{u}\|_2^2\ . \tag{14}$$

This estimate, however, is obviously not going to be tight for small $\mathsf{Re}$, when the flow is not very turbulent. To improve this estimate so that it take into account the fact that for small $\mathsf{Re}$ the energy does not "propagate" much from the large to the small wavenumbers, we split the velocity $\mathbf{u}$ into a "low-$k$" component, $\mathcal{P}\mathbf{u}$, and a "high-$k$" one, $\mathbf{u}-\mathcal{P}\mathbf{u}$. We will still use the uniform estimate (13) as well as the inequality $ab \le \frac{1}{2}(za^2 + \frac{1}{z}b^2)$ which holds for any $z > 0$:

$$\begin{aligned}\left|\int [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\frac{\mathbf{p}}{\|\mathbf{p}\|_2}\,\mathrm{d}^d\mathbf{x}\right| &= \left|\int [\mathbf{p}+(\mathbf{u}-\mathbf{p})]\cdot\frac{\nabla\mathbf{p}}{\|\mathbf{p}\|_2}\cdot[\mathbf{p}+(\mathbf{u}-\mathbf{p})]\,\mathrm{d}^d\mathbf{x}\right| \\ &\le \frac{2\pi\sqrt{d}}{\ell^{1+(d/2)}}\int \left(2\,|\mathbf{p}|\,|\mathbf{u}-\mathbf{p}|+|\mathbf{u}-\mathbf{p}|^2\right)\,\mathrm{d}^d\mathbf{x} \\ &\le \frac{2\pi\sqrt{d}}{\ell^{1+(d/2)}}\int \left[z|\mathbf{p}|^2+\left(\tfrac{1}{z}+1\right)\,|\mathbf{u}-\mathbf{p}|^2\right]\,\mathrm{d}^d\mathbf{x} \\ &\le \frac{2\pi\sqrt{d}}{\ell^{1+(d/2)}}\left[\left(\tfrac{1}{z}+1\right)\,\|\mathbf{u}\|_2^2+\left(z-\tfrac{1}{z}-1\right)\|\mathbf{p}\|_2^2\right]\;. \end{aligned} \tag{15}$$

Putting together (14) and (15), we find

$$\begin{aligned}\ell^d\epsilon\frac{1}{\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2} &\le \frac{1}{2}\frac{\mathrm{d}\,\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2}{\mathrm{d}t} \\ &\quad+\frac{2\pi\sqrt{d}}{\ell^{1+(d/2)}}\,\min\left\{\|\mathbf{u}\|_2^2,\;\left(\tfrac{1}{z}+1\right)\|\mathbf{u}\|_2^2+\left(z-\tfrac{1}{z}-1\right)\|\mathcal{P}\mathbf{u}\|_2^2\right\} \\ &\quad+\nu\,\frac{4\pi^2}{\ell^2}\,\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2\;. \end{aligned} \tag{16}$$

Now take the time average of all terms in the above inequality. First note that the average of the time derivative of $\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2$ gives zero due to the boundedness of $\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2$ (which follows from the boundedness of $\|\mathbf{u}(\cdot,t)\|_2$; see (11)). To estimate the other terms, we use Jensen's inequality: if a function $\theta$ is convex and $\langle\cdot\rangle$ stands for averaging, then $\langle\theta\circ g\rangle\ge\theta\left(\langle g\rangle\right)$ for any real-valued function $g$. Applying this inequality to the case $g(t)=\|\mathcal{P}\mathbf{u}(\cdot,t)\|_2$ and the convex function $\theta(t)=t^2$, we obtain (same as Cauchy-Schwarz)

$$\langle\|\mathcal{P}\mathbf{u}\|_2\rangle\le\sqrt{\langle\|\mathcal{P}\mathbf{u}\|_2^2\rangle}=\ell^{d/2}\,V\;.$$

On the other hand, if we take $\theta(t)=\frac{1}{t}$ for $t>0$, we deduce

$$\left\langle\frac{1}{\|\mathcal{P}\mathbf{u}\|_2}\right\rangle\ge\frac{1}{\langle\|\mathcal{P}\mathbf{u}\|_2\rangle}\ge\frac{1}{\sqrt{\langle\|\mathcal{P}\mathbf{u}\|_2^2\rangle}}=\frac{1}{\ell^{d/2}\,V}\;.$$

Plugging these estimates into (16), we obtain

$$\epsilon\le\nu\frac{4\pi^2}{\ell^2}V^2+\frac{2\pi\sqrt{d}}{\ell}\,\min\left\{U^2V,\;\left(\tfrac{1}{z}+1\right)U^2V+\left(z-\tfrac{1}{z}-1\right)V^3\right\}\;.$$

In terms of the non-dimensional energy dissipation rate (6), we can rewrite this inequality in the form

$$\beta\le\frac{4\pi^2}{\mathsf{Re}}\,\mathfrak{p}^2+2\pi\sqrt{d}\,\phi(\mathfrak{p},z)\;, \tag{17}$$

where we have introduced the function

$$\phi(\mathfrak{p},z):=\min\left\{\mathfrak{p},\;\left(\tfrac{1}{z}+1\right)\mathfrak{p}+\left(z-\tfrac{1}{z}-1\right)\mathfrak{p}^3\right\}\;. \tag{18}$$

### 2.3 Compatibility of the lower and upper bounds on $\beta$

Assembling the lower and upper bounds (12) and (17), we have

$$\frac{4\pi^2}{\mathsf{Re}}(2-\mathfrak{p}^2) \leq \beta \leq \frac{4\pi^2}{\mathsf{Re}}\,\mathfrak{p}^2 + 2\pi\sqrt{d}\,\phi(\mathfrak{p},z)\ . \tag{19}$$

The compatibility of the two bounds on $\beta$ imposes restrictions on the allowed range of $\mathfrak{p}$, namely, $\mathfrak{p}$ should satisfy the inequality

$$\mathfrak{p}^2 + \frac{\sqrt{d}\,\mathsf{Re}}{4\pi}\,\phi(\mathfrak{p},z) - 1 \geq 0\ . \tag{20}$$

In the interval $\mathfrak{p} \in [0,1]$, this inequality is satisfied for $\mathfrak{p} \in [p_{\min}(\mathsf{Re},z),1]$, where $p_{\min}(\mathsf{Re},z) \approx \frac{4\pi}{\sqrt{d}\,\mathsf{Re}}$ for large $\mathsf{Re}$. Clearly, the lower bound on the range of $\mathfrak{p}$ is more meaningful for smaller $\mathsf{Re}$.

### 2.4 Optimizing the upper bound on $\beta$

Since we do not have *a priori* control over $\mathfrak{p}$, we will derive an upper bound for $\beta$ by maximizing the upper bound in (19) over $\mathfrak{p}$, after which we use the freedom in the choice of the parameter $z > 0$ to minimize for any given $\mathsf{Re}$, which results in

$$\beta \leq \min_{z>0}\ \max_{\mathfrak{p}\in[p_{\min}(\mathsf{Re},z),1]} \left[\frac{4\pi^2}{\mathsf{Re}}\,\mathfrak{p}^2 + 2\pi\sqrt{d}\,\phi(\mathfrak{p},z)\right]\ . \tag{21}$$

Although this procedure is not difficult to implement numerically, we will analyze only the case of high $\mathsf{Re}$ where the analysis can be carried out analytically. First notice that for high $\mathsf{Re}$, the lower bound $p_{\min}(\mathsf{Re},z)$ is very small, so the maximization over $\mathfrak{p}$ can be taken in the entire interval $[0,1]$. Thus $\phi(\mathfrak{p},z) \leq \phi^*(z) := \max_{\mathfrak{p}\in[0,1]} \phi(\mathfrak{p},z) = (1+z-z^2)^{-1/2}$ for $z \in [0, \frac{1+\sqrt{5}}{2})$. Since for large $\mathsf{Re}$ the $\mathsf{Re}$-independent term in the right-hand side of (21) is dominating, we have the high-$\mathsf{Re}$ estimate

$$\beta \leq \min_{z\in[0,\frac{1+\sqrt{5}}{2})} \left[\frac{4\pi^2}{\mathsf{Re}}\,\phi^*(z)^2 + 2\pi\sqrt{d}\,\phi^*(z)\right] = \frac{16\pi^2}{5\mathsf{Re}} + \frac{4\pi\sqrt{d}}{\sqrt{5}}\ . \tag{22}$$

At high $\mathsf{Re}$, the value or $\mathfrak{p}$ maximizing $\phi(\mathfrak{p},z)$ is $\frac{2}{\sqrt{5}}$. We remark that it is not difficult to prove that the upper bound (22) is optimal (i.e., coincides with (21)) for $\mathsf{Re} \geq \frac{8\pi}{3\sqrt{5d}}$.

## 3 Discussion

In dimension 3, the scaling of the upper bound is in accord with conventional turbulence theory: at high $\mathsf{Re}$, $\epsilon \sim \frac{U^3}{\ell}$ is independent of the molecular viscosity. For the type of forcing considered here, we find $\beta \leq 4\pi\sqrt{\frac{3}{5}} \approx 9.7339\ldots$.

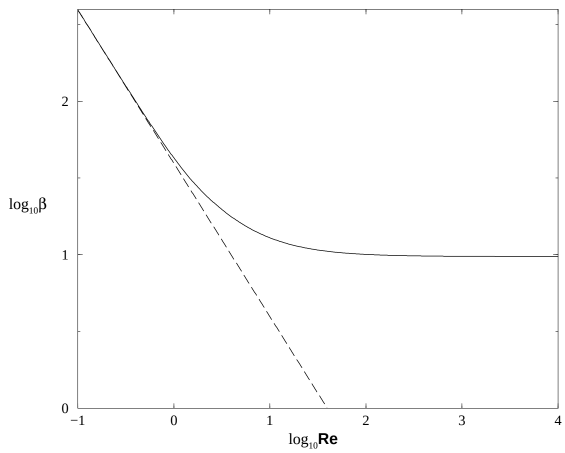


**Fig. 1.** Upper and lower bounds on $\beta$ (solid and dashed lines, resp.)

A plot of the bounds is presented in Figure 1. At low $\mathsf{Re}$, the upper and lower bounds converge to each other. While it is difficult to compare these bounds quantitatively with DNS results, we note from [7] that at high $\mathsf{Re}$, values of $\beta$ are typically around 1. Hence, our rigorous analysis, while yielding the expected scaling, overestimates the constants by about an order of magnitude.

In the 3-dimensional case, if we assume that the cascade is Kolmogorov, i.e., the spectral density of the energy is given by $E_{\mathrm{K}}(k) = C\epsilon^{2/3}k^{-5/3}$, we can estimate the "Kolmogorov" value $\mathfrak{p}_{\mathrm{K}}$ as follows:

$$E_{\mathrm{kin,\ total}} \approx \int_{2\pi/\ell}^{\infty} E_{\mathrm{K}}(k)\,\mathrm{d}k\ , \quad E_{\mathrm{kin,\ low}\,k} \approx \frac{2\pi}{\ell}\,E_{\mathrm{K}}\left(\frac{2\pi}{\ell}\right)\ ,$$

which gives $\mathfrak{p}_{\mathrm{K}} \approx \sqrt{\frac{2}{3}}$. Plugging this value in (21) and minimizing over $z$, we obtain the (approximate) estimate

$$\beta \leq \frac{8\pi^2}{3\,\mathsf{Re}} + 2\sqrt{2}\pi \approx \frac{26.3}{\mathsf{Re}} + 8.9\ ,$$

which gives a slight improvement compared with the bounds (22).

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