MATH 4103 Additional problem assigned on 2/23/16

Additional problem. Show that $u(x,y)$ is harmonic in some domain, and find a harmonic conjugate $v(x,y)$ when:

(a) $u(x,y) = y^3 - 3x^2y$;

(b) $u(x,y) = \sinh x \, \sin y$;

(c) $u(x,y) = 2x - x^3 + 3xy^2$ (solved below; no need to copy the solution).

Hint to part (b): $\frac{\mathrm{d}}{\mathrm{d}x} \sinh x = \cosh x$, $\frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sinh x$.

Solution of part (c): Checking that $u(x,y)$ is a harmonic function by direct computation:

$$u_x = 2 - 3x^2 + 6y^2 \ , \quad u_{xx} = -6x \ ; \quad u_y = 6xy \ , \quad u_{yy} = 6x \ ; \quad u_{xx} + u_{yy} = -6x + 6x = 0 \ .$$

The functions u and v must satisfy the Cauchy-Riemann equations, $u_x = v_y$, $u_y = -v_x$. Therefore, the function $v(x,y)$ must satisfy the equations

$$v_x = -u_y = -6xy \ , \tag{1}$$

$$v_y = u_x = 2 - 3x^2 + 3y^2 \ . \tag{2}$$

First we integrate (1) with respect to x (while treating y as a constant):

$$v(x,y) = \int v_x(x,y) \, \mathrm{d}x = \int (-6xy) \, \mathrm{d}x = -3x^2y + \phi(y) \ , \tag{3}$$

where ϕ is a (smooth) function of one variable, which we will determine in a second. Having established that $v(x,y)$ has the form given by the right-hand side of (3), we plug this in equation (2) to find the unknown function ϕ:

$$v_y(x,y) = \frac{\partial}{\partial y} \big(-3x^2y + \phi(y) \big) = -3x^2 + \phi'(y) = 2 - 3x^2 + 3y^2 \ .$$

This implies that

$$\phi'(y) = 2 + 3y^2 \ ,$$

which integrates to

$$\phi(x,y) = 2y + y^3 + C \ ,$$

where C is a "genuine" constant (i.e., a constant that does not depend on x and on y). This yields

$$v(x,y) = -3x^2y + \phi(y) = -3x^2y + 2y + y^3 + C \ . \tag{4}$$

Note that we could have done the calculation in different order: first integrating (2):

$$v(x, y) = \int v_y(x, y)\,\mathrm{d}y = \int \left(2 - 3x^2 + 3y^2\right)\,\mathrm{d}y = 2y - 3x^2y + y^3 + \psi(x) \ , \tag{5}$$

where ψ is a (smooth) function of one variable, to be determined. To find ψ, plug $v(x, y)$ from (5) into (1):

$$v_x(x, y) = \frac{\partial}{\partial x}\left(2y - 3x^2y + y^3 + \psi(x)\right) = -6xy + \psi'(x) = -6xy$$

so that $\phi'(x) = 0$, i.e., $\psi(x) = C$ (a "genuine" constant). This, together with (5), implies

$$v(x, y) = 2y - 3x^2y + y^3 + \psi(x) = 2y - 3x^2y + y^3 + C \ ,$$

which, naturally, is the same result as (4).