Additional problem (this is NOT an FFT problem!)

Additional problem. In this problem you will generalize the results about Gabriel's horn (shown in Exercise 8.2/25). Consider the region

$$\mathscr{R}_\alpha = \left\{ (x,y) \,\middle|\, x \geq 1,\ 0 \leq y \leq \frac{1}{x^\alpha} \right\} \ ,$$

in the (x, y)-plane, where α is a positive constant (not necessarily integer). Let $\mathscr{D}_\alpha$ be the solid of revolution obtained by rotating $\mathscr{R}_\alpha$ about the x-axis.

(a) Find the range of values of α for which the volume of $\mathscr{D}_\alpha$ is finite, by using the method of slicing $\mathscr{D}_\alpha$ by parallel planes (as in Section 5.2). Find the volume of $\mathscr{D}_\alpha$ for the values of α for which this volume is finite.

(b) Find the range of values of α for which the volume of $\mathscr{D}_\alpha$ is finite, by using the method of cylindrical shells (as in Section 5.3). Find the volume of $\mathscr{D}_\alpha$ for the values of α for which this volume is finite. (Of course, the result should be the same as in part (a), but obtained in a different way.)

(c) Set up the integral for the (side) surface area of $\mathscr{D}_\alpha$. Show that for $\alpha > 1$ the (side) surface area is finite. Do *not* attempt to solve the integral, use the Comparison Theorem.

(d) Use the Comparison Theorem to show that for $\alpha < 1$ the (side) surface area is infinite.