Some methods for approximate computation of definite integrals

Goal: Compute approximately the numerical value of the integral

$$I_{\text{exact}} = \int_a^b f(x)\,dx \ ,$$

for a given function f and finite numbers a, b ("finite" means that they are not ∞ or $-\infty$).

Notations. Let n be a natural number (i.e., a positive integer), and x_j (with $j = 0, 1, \ldots, n$) are numbers such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \ .$$

For simplicity, we assume that the numbers x_i are equidistant, i.e., that

$$x_i - x_{i-1} = \Delta x = \frac{b-a}{n} \qquad \text{for every } i = 0, 1, \ldots, n \ .$$

With this choice, we have $x_i = a + i\,\Delta x$, $i = 0, 1, \ldots, n$.

Simplest methods:

- left Riemann sums: $\displaystyle L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$;
- right Riemann sums: $\displaystyle R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$;
- midpoint rule: $\displaystyle M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$;
- trapezoidal rule: $\displaystyle T_n = \left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right] \frac{\Delta x}{2}$;
- Simpson's rule (the integer n must be even):

$$S_n = \left[f(x_0) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + f(x_n) \right] \frac{\Delta x}{3} \ .$$

Errors of the different methods: Let $E_n = |I_{\text{approx}} - I_{\text{exact}}|$ be the (absolute) error of a method, then

- the errors of the left and right Riemann sums behave like $C\,(\Delta x)$;
- the errors of the midpoint and the trapezoidal rules behave like $C\,(\Delta x)^2$;
- the error of the Simpson's rule behaves like $C\,(\Delta x)^4$.

A numerical example: In the Mathematica notebook

`http://www2.math.ou.edu/~npetrov/illustration-approximate-integration-methods.nb` a printout of which can be found at

`http://www2.math.ou.edu/~npetrov/illustration-approximate-integration-methods.pdf`

The approximate values of the integral

$$I_{\text{exact}} = \int_4^9 \sqrt{x}\, dx = \frac{38}{3}$$

has been computed by using each of the above methods for approximate integration, for $n = 10$, 100, 1000, 10000, and 100000. The (absolute) errors are displayed in the table below.

n	$\lvert L_n - I_{\text{exact}}\rvert$	$\lvert R_n - I_{\text{exact}}\rvert$	$\lvert M_n - I_{\text{exact}}\rvert$	$\lvert T_n - I_{\text{exact}}\rvert$	$\lvert S_n - I_{\text{exact}}\rvert$
10	2.51735×10^{-1}	2.48264×10^{-1}	8.67285×10^{-4}	1.73523×10^{-3}	3.47115×10^{-6}
100	2.50173×10^{-2}	2.49826×10^{-2}	8.68047×10^{-6}	1.73610×10^{-5}	3.53252×10^{-10}
1000	2.50017×10^{-3}	2.49982×10^{-3}	8.68055×10^{-8}	1.73611×10^{-7}	3.53316×10^{-14}
10000	2.50001×10^{-4}	2.49998×10^{-4}	8.68055×10^{-10}	1.73611×10^{-9}	3.53317×10^{-18}
100000	2.50000×10^{-5}	2.49999×10^{-5}	8.68055×10^{-12}	1.73611×10^{-11}	3.53317×10^{-22}

Note how the errors decrease as n increases by a factor of 10, and therefore Δx decreases by a factor of 10, for each of the methods. Compare this numerical observation with the theoretical results on the errors of the different methods given above.