MATH 5403 – Calculus of Variations

Fall 2023, TR 1:30–2:45 p.m., 404 PHSC

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Textbook: I. M. Gelfand, S. V. Fomin, *Calculus of Variations*, Dover, 1991, $11.79 on Amazon.

We will also use parts of the following books, freely available online for OU students:

- B. van Brunt, *The Calculus of Variations*, Springer, 2004,
- A. Rojo, A. Bloch, *The Principle of Least Action*, Cambridge University Press, 2018,
- H. Kielhöfer, *Calculus of Variations: An Introduction to the One-Dimensional Theory with Examples and Exercises*, Springer, 2018,
- P. Freguglia, M. Giaquinta, *The Early Period of the Calculus of Variations*, Birkhäuser, 2016.

Prerequisites: The formal prerequisites are MATH 4433 (Intro to Analysis I) or MATH 3423 (Physical Math II) or MATH 4163 (PDE), but a good understanding of multivariable calculus and perhaps some exposure to elementary PDEs would be enough. No background in physics is expected.

General description: Calculus of variations is a field of analysis that studies maxima and minima of functionals. Usually functionals of interests are defined as integrals of certain functions, and the goal is to study extremals of such functionals (i.e., functions that minimize or maximize the value of the functional) – their existence, uniqueness, regularity, methods for constructing extremals. Calculus of variations was motivated by physics and has many practical applications.

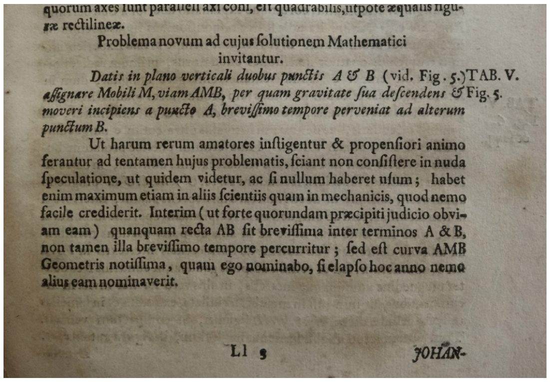
Tentative lists of topics:

- Functionals: definition, properties, weak and strong extrema of functionals.
- The first variation: Euler-Lagrange equations, invariance of the Euler-Lagrange equations, Du Bois-Reymond's lemma, particular cases.
- Generalizations: higher-order derivatives, multiple unknown functions, multiple independent variables.
- Constrained systems: Lagrange multipliers, isoperimetric constraints, holonomic constraints.
- The second variation: Legendre and Jacobi conditions, Jacobi fields, conjugate points.
- Variable endpoints: transversality, broken extremals, Weierstrass-Erdmann conditions.
- Canonical formalism: phase space, Hamilton's equations, Poisson brackets, Hamiltonian flow, canonical transformations, Legendre transform.
- Hamilton-Jacobi theory: first-order PDEs, characteristic equations, Hamilton-Jacobi equation, application to geometric optics – Fermat's principle, Huygens construction.
- Conservation laws: local one-parameter transformation groups, generators, variational symmetries, Noether's Theorem.
- Sufficient conditions for a strong extremum: field of extremals, Hilbert invariant integral, Weierstrass $\mathscr{E}$-function.

We will cover many interesting examples from geometry (geodesics, Queen Dido's isoperimetric problem, minimal surfaces), mechanics (particle motion, curve of steepest descent, shape of a hanging cable, motion of strings, membranes, and fluids), and optics (reflection, refraction, caustics).

In this class students will gain a deeper understanding of some subtle concepts of calculus and their connection with physics. No background in physics is expected. Although this is a graduate Math class, no background in graduate level Math is needed, a good command of multivariable calculus is enough. We will review briefly some concepts of calculus as needed (Taylor series, integral theorems of calculus, changes of variables, implicit function theorem, elementary facts about function spaces). The course should be of interest not only for Mathematics students, but also to students majoring in Physics, Meteorology, or Engineering.

Calculus of Variations originated in a challenge to mathematicians around the world by Johann Bernoulli published in the journal *Acta Eruditirum* in June 1696—see the figure below.



quorum axes ſunt paralleli axi coni, eſt quadrabilis, utpote æqualis figuræ rectilineæ.

Problema novum ad cujus ſolutionem Mathematici invitantur.

Datis in plano verticali duobus punctis A & B (vid. Fig. 5.) TAB. V. Fig. 5. *aſſignare Mobili M, viam AMB, per quam gravitate ſua deſcendens & moveri incipiens a puncto A, breviſſimo tempore perveniat ad alterum punctum B.*

Ut harum rerum amatores inſtigentur & propenſiori animo ferantur ad tentamen hujus problematis, ſciant non conſiſtere in nuda ſpeculatione, ut quidem videtur, ac ſi nullum haberet uſum; habet enim maximum etiam in aliis ſcientiis quam in mechanicis, quod nemo facile crediderit. Interim (ut forte quorundam præcipiti judicio obviam eam) quanquam recta AB ſit breviſſima inter terminos A & B, non tamen illa breviſſimo tempore percurritur; ſed eſt curva AMB Geometris notiſſima, quam ego nominabo, ſi elapſo hoc anno nemo alius eam nominaverit.

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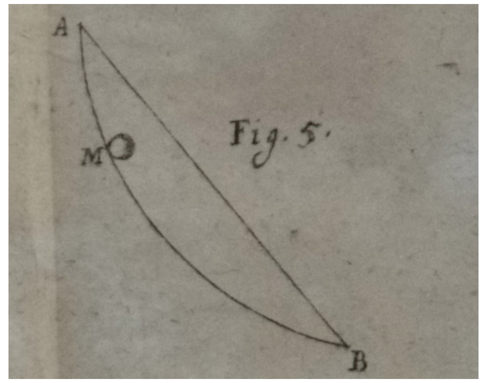

Figure 1: Johann Bernoulli's challenge and the figure accompanying it.

Here is a rough translation of the challenge.

New Problem Which Mathematicians Are Invited to Solve

Given the two points A & B in a vertical plane (see Fig. 5.), assign to the mobile particle M the path AMB, along which it descends moved by gravity, starting from point A, in the shortest time to reach the second point B.

To arouse in lovers of such things the desire to undertake the solution of this problem, it may be pointed out that the question proposed does not, as might appear, consist of mere speculation having therefore no use. On the contrary, as no one would readily believe, it has great usefulness in other branches of science such as mechanics. Meanwhile (to forestall hasty judgement) [it may be remarked that] although the straight line AB is indeed the shortest between the points A and B, it nevertheless is not the path traversed in the shortest time. However the curve AMB, whose name I shall give if no one else has discovered it before the end of this year, is one well known to geometers.

A year later solutions were published by Johann Bernoulli, by his brother Jacob Bernoulli, a short note by Leibniz saying that he would not publish his solution because it was similar to that of the Bernoulli brothers, and a correct answer—but no solution!—by an anonymous author (who was in fact Newton).

The History of Science Collection on the 5th floor of Bizzell Library has physical copies of these journals which you can use in the library! Of course, you have to know Latin to read the journal.

Figure 2: The June 1696 issue of *Acta Eruditorum* in the History of Science Collection in Bizzell.