Mechanics

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Synonyms

N/A

Main Text

Mechanics is an area of physics that studies the motions of material objects. The topic of this article *is classical mechanics*, i.e., mechanics that does not use theory of relativity and quantum theory. Although classical mechanics does not describe reality adequately for very large speeds or very small distances, it is an active area of physics, with important recent developments and innumerable practical applications. Moreover, ideas that were first developed within the framework of classical mechanics turned out to be very fruitful in many other areas of physics. In this article we focus on some important concepts of classical mechanics, presenting them in the chronological order of their introduction. For a detailed account of the history of classical mechanics until the 1950's we recommend the monograph [1].

Pre-Newtonian physics

The roots of classical mechanics are in antiquity. It flourished in ancient Greece, in particular, in the works of Aristotle (384-322 BC). Aristotelian scientific doctrines had profound influence on the scientific thought until the times of Galileo and Newton. Ancient scientists discovered some simple mechanical laws (i.e., the law of buoyancy), realized that the Earth had a spherical shape and were able to measure its radius.

The modern development of classical mechanics started in the 1400's and was related with the name of Nicolaus Copernicus (1473-1543), who challenged the geocentric planetary system of Aristotle and Ptolemy (Claudius Ptolemaeus, c. 90-c. 168 AD). Free of the stifling influence of Aristotelian ideas, observational astronomy developed quickly, culminating in the experimental discovery of the three laws of planetary motion of Johannes Kepler (1571-1630). In the works of Galileo Galilei (1564-1642), René Descartes (1596-1650), and Christiaan Huygens (1629-1695), reliance on observational or experimental evidence were emphasized rather than pure reasoning.

Quantitative period of classical mechanics

The foundations of our present understanding of classical mechanics were laid by Isaac Newton (1643-1727) in his magisterial treatise *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"), originally published in 1686; see the annotated edition [2]. Using the three laws of mechanics formulated in *Principia* together with his *law of gravitation*, Newton was able to derive the laws of Kepler.

To describe the rate of change of physical quantities, Newton developed differential and integral calculus (at the same time as, and independently of, Gottfried Leibniz (1646-1716))—a mathematical theory of enormous importance. In Newtonian mechanics, the motion of an object is governed by ordinary differential equations that express its acceleration (the rate of change of its velocity) in terms of the mass of the object and the forces acting on it. The entire subsequent development of classical mechanics is based on the understanding of the mathematical structures behind Newton's theory and its relation with other areas of mathematics and physics.

At the very foundation of Newton's theory of mechanics are such fundamental concepts as *space*, *time*, *force*, *matter*, *mass*. As it is not possible to discuss here these notions in depth, we only direct the reader to [3], and proceed to the post-Newtonian development of classical mechanics.

Variational principles and first steps towards geometrization of mechanics

Pierre de Fermat (1601-1665) showed that Snell's law of refraction of light can be derived from the postulate that, when traveling from one point to another, a light ray follows the path that minimizes the travel time. Pierre-Louis de Maupertuis (1698-1759) suggested that in the realm of mechanical phenomena the situation is similar—namely, there exists certain quantity, called *action*, that is minimized in mechanical processes. He interpreted this as a manifestation of the wisdom of God, who does everything in the most efficient way. The vague (and mathematically incorect) ideas of Maupertuis were put on a firm mathematical ground by Leonard Euler (1707-1783), Joseph-Louis Lagrange (1736-1813), William Hamilton (1805-1865), and Carl Jacobi (1804-1851). Given the initial and the final configurations of the mechanical system (i.e., the initial and final positions and velocities of all particles constituting the system), the *variational principle* states that the evolution of the system between the initial and the final moments of time is such that the action of the system is extremal (that is, minimal or maximal). The action is expressed through a function of the time and the current positions and velocities of the particles called the *Lagrangian* of the system. From the fact that the evolution of the system is an extremum of the action, one can derive ordinary differential equations, called the *Euler-Lagrange equations*, governing the temporal evolution of the system. If one starts with the Lagrangian of a physical system, the corresponding Euler-Lagrange equations are the Newton's equations of motion of the system.

Lagrange's description of a physical system has several advantages over the description of Newton. For the working physicist, writing down the action is usually easier than writing down Newton's equations. The symmetries of the system (see below) are more clearly visible from the Lagrangian of the system rather than from Newton's equations. Perhaps most importantly, Lagrange's approach relies heavily on the geometry of the space of coordinates and velocities of the system.

Lagrange's formalism is especially useful if the system is constrained. A simple example of a constrained system is two point particles connected by a rod of constant length; the whole system is free to move in three-dimensional space. Since each particle has three coordinates, in the absence of the rod the position of the system would be described by the total of six functions of time (giving all the coordinates of the particles at each moment of time). The rod, however, imposes the constraint that the distance between the particles should be the same at each moment. Because of this constraint, the position of the particles is described by only five functions of time.

Hamilton's equations and symplectic geometry

Another big step in the interpretations of the laws of classical mechanics was their reformulation as the so-called *canonical equations* of Hamilton. Besides their inherent importance for solving concrete physical problems, these equations provided a completely new way of thinking about the geometry behind the dynamics of the system. In this formalism, the space where the evolution of a mechanical system takes place is the so-called *phase space*—a space of even dimension where half of the dimensions give the position and the other half give the corresponding *momenta* of the particles (in high school physics the momentum of a point object is the product of its mass and its velocity, but in general the definition of momentum is more complicated). Hamilton's canonical equations then imply that the phase space of a physical system is a *symplectic manifold*, i.e., an even-dimensional space endowed with a special geometric structure called a *symplectic form*. The state of the physical system at a given moment of time corresponds to a point in this space, and the temporal evolution of the system with time is described by the motion of this point. A *phase space trajectory* is a curve in the phase space of a physical system that describes the evolution of the system with time (from mathematical point of view, a phase space trajectory is a solution of the Hamilton's canonical equations). For each point in the phase space, there is only one phase space trajectory passing through it. This geometric description flourished in the 19th and 20th centuries, and is the theoretical foundation of our present understanding of classical mechanics. For details and references see the modern overview [4] and the growing body of online information *Scholarpedia* [5].

Symmetries and conservation laws

Among the most important ideas in mechanics (and in physics in general) is the connection of the symmetries of the system with the existence of *conserved quantities*, i.e., quantities that do not change in time (see the contribution of Belot in [3]). For example, from the plausible assumption that time is homogeneous (i.e., that the evolution of a physical system from certain initial conditions does not depend on the initial moment when we let the system evolve), one can derive the law of conservation of energy. These ideas were generalized in early 20th century by Emmy Noether (1882-1935) who formulated a general mathematical theorem relating symmetries and conservation laws. Together with the modern geometric methods of mechanics (especially taking into account the symplectic geometry behind Hamilton's equations) and an area of mathematics called "group theory," these ideas provide a method for reduction of physical systems with symmetry. The reduction procedure eliminates the redundant coordinates from the original phase space of the system and replaces the original phase space with a phase space with smaller dimension which describes the evolution of the system in the most efficient way (see [4], Chapter 3).

Theory of dynamical systems: a neo-qualitative period of classical mechanics

A radically new point of view on classical mechanics was developed in the studies of Henri Poincaré (1854-1912) culminating in his 3-volume magnum opus *Les méthodes nouvelles de la mécanique céleste* ("New Methods of Celestial Mechanics"), published in the last decade of the 19th century. Instead of concentrating on one particular phase space trajectory, Poincaré proposed to study the behavior of all possible phase space trajectories of the physical system, i.e., all possible evolutions of the system starting with all allowed initial conditions. The family of all phase space trajectories in the phase space of the system is called the *phase portrait* of the system [5]. Within this approach, one in interested in questions like "Does the system have periodic solutions?", or "Are the velocities of the particles bounded?", or the important question "If the system is perturbed slightly, does the behavior of its solutions differ significantly from the behavior of the solutions of the original system?"

Poincaré realized the importance of such qualitative questions, and understood that the answer to such questions is intimately related to the global geometric properties of the phase space of the system. These new ideas were advanced significantly in the 1920's by George Birkhoff (1884-1944).

Poincaré and Birkhoff enriched mechanics with new goals and new tools; the area of physics and mathematics created in their works was called *theory of dynamical systems*. This theory emphasized the importance of the qualitative theory of differential equations—a field of mathematics virtually nonexistent before that. Instead of solving the equations explicitly, theory of dynamical systems is interested in the behavior of whole sets of trajectories in the phase space, in particular in their stability. In connection with this, it studies the *bifurcations* of the system, i.e., the abrupt changes in the phase portrait caused by changes of the values of the parameters of the system. The qualitative features of the phase portrait of the system are closely related to the existence of certain geometric objects in the phase space, e.g., *invariant manifolds*. An invariant manifold is a surface in the phase space of the system such that each phase space trajectory starting from a point on this surface stays on the same surface forever. Clearly, the presence of an invariant manifold is an obstruction to the possible qualitative behavior of the system because phase space trajectories are not allowed to cross an invariant manifold.

Another fundamental innovation in theory of dynamical systems is that, since it studies the behavior of whole sets of phase space trajectories, it has to employ probabilistic concepts in dealing with the physical system. This makes a connection with *statistical mechanics*—an area of physics that studies the properties of systems consisting of a very large number of particles, so that the particles cannot be described individually (examples of such systems are gas in a container or a crystal). Poincaré and Birkhoff were among the creators of a new field of physics and mathematics called *ergodic theory*, whose main object of interest is the evolution of the probabilities related to the description of the system [5]. The probabilistic approach is also related with *information theory* which studies the quantification of information, and is of great importance for modern communication systems.

Further developments in the theory of dynamical systems

Initially created by Poincaré within the framework of celestial mechanics, theory of dynamical systems evolved quickly, often stimulated by the rapid development of science and technology in the 20th century. For a riveting popular history of dynamical systems see [6], and a scholarly analysis can be found in [7].

In the early days of radio engineering, Balthasar van der Pol (1889-1959) noticed strange irregular noises in experiments with radio circuits, which motivated mathematical studies by Mary Cartwright (1900-1998), John Littlewood (1885-1977) and Norman Levinson (1912-1975) in the 1940s. These studies culminated in the discovery by Stephen Smale (b. 1930) in the mid-1960's of a structure that is at the heart of modern theory of dynamical systems, which is now called the *Smale horseshoe* [5]. If a dynamical system has a horseshoe, then it exhibits complicated behavior ("chaos").

A fundamental mathematical result on the behavior of Hamiltonian dynamical systems (i.e., dynamical systems described by Hamilton's canonical equations) is the celebrated *KAM theorem* named after Andrey Kolmogorov (1903-1987), Vladimir Arnold (b. 1937), and Jürgen Moser (1928-1999), who proved it in the 1960s. It is concerned with the persistence of certain types of motions in a completely integrable Hamiltonian system under small perturbations. The behavior of the phase space trajectories in a completely integrable Hamiltonian system is very orderly—each phase space trajectory belongs to a torus (a particular kind of a surface) in the phase space of the system. The KAM theorem states roughly that, under suitable assumptions, most of these invariant tori survive (and are only slightly deformed) when the system is slightly perturbed, hence the "order" in an integrable system is not destroyed completely by small perturbations, but instead the "disorder" occurs in the system gradually as the perturbation becomes stronger [5].

The advent of modern computing devices influenced deeply the development of modern physics. One of the first computers, MANIAC I, was used in the early 1950s by Enrico Fermi (1901-1954), John Pasta (1918-1984), and Stanislaw Ulam (1909-1984), who studied numerically a one-dimensional chain of particles linked by springs (where the springs behaved slightly nonlinearly)—a one-dimensional analogue of atoms in a crystal. The results of their numerical simulations were strikingly different from what the scientists expected, thus showing that some very foundational ideas in physics (related to ergodic theory) had to be reconsidered [5].

Another remarkable discovery came from meteorology in 1963, with the publication of the famous paper "Deterministic nonperiodic flow" of Edward Lorenz (1917-2008). Lorenz modeled on a computer the phenomena in the atmosphere when it is heated from below by the Earth surface. He used a system of three nonlinear differential equations that have the feature that, due to the dissipation in the system (i.e., the loss of energy due to the viscosity of the air), as time passes, a three-dimensional domain tends with time to a set of lower dimension called an *attractor*. From the numerical studies of the attractor in the Lorenz system, it looked like it is a set with a complicated geometric structure.

The dimension of the attractor seemed to be greater than two but smaller than three; such objects were called *strange attractors* (the word "strange" is sometimes used in this context with a different meaning). The concept of objects of non-integer dimension was studied in pure mathematics since the early 20th century, notably by Felix Hausdorff (1868-1942). The presence of an attractor in the Lorenz system of differential equations made it clear that objects of fractional dimensions are non only a mathematical curiosity, but occur naturally in physical systems. Usually such objects coming from physics reveal *self-similarity*, i.e., a portion of the object looks approximately like the whole object under an appropriate magnification. The ubiquity of such object in natural phenomena was brought to the mainstream mathematics and physics by Benoît Mandelbrot (b. 1924), who named them *fractals*. Today fractals are commonplace in natural sciences and engineering, and there are many concepts of dimension that are useful in different contexts [5].

It turned out that the Lorenz attractor has also the property that the distance between two nearby points on the attractor grows exponentially fast with time, or, in technical terms, exhibits *sensitive dependence on initial conditions*. This sensitive dependence—metaphorically called the *butterfly effect* by Lorenz [5]—gives an idea why there are fundamental difficulties in long-term weather prediction.

The attractor observed numerically by Lorenz turned out to resist rigorous mathematical analysis, so John Guckenheimer (b. 1945) introduced a "geometric model" for the Lorenz equations—a system that exhibits the same features as the ones observed numerically for the Lorenz equations, but is simpler and hence amenable to rigorous analysis. The existence of an attractor in such geometric models was proved rigorously, and its properties were studied extensively. However, the question whether the attractor in the original Lorenz equations indeed exists remained open. In 1997 Smale posed this question as one of the 18 "mathematical problems for the next century". The question was answered positively the very next year by Warwick Tucker (b. 1970), who also proved rigorously that the attractor is robust, i.e., it is not destroyed by small changes in the coefficients in the Lorenz system. Tucker's proof was *computer-assisted*, i.e., it used a computer for numerical computations, but—unlike in the traditional way computers are used—kept track of all possible numerical errors, which made his proof mathematically rigorous. This episode provides an interesting example of the interactions of modern mathematics and computing.

Determinism in classical mechanics

Newton's laws are evolution equations, i.e., if one knows the masses, positions, and velocities of all objects at some moment, as well as the forces of interaction between the objects, the whole subsequent motion of the bodies is completely determined (i.e., can be calculated with arbitrary accuracy). The uniqueness of the evolution in classical mechanics is the basis of the determinism of Pierre-Simon Laplace (1749-1827). If the world is governed by the laws of classical mechanics, then Laplace's determinism holds and there is no free will. There are, however, some situations when the equations of classical mechanics admit non-unique solutions, for example when the force is proportional to the square root of the velocity (see the example of Hutchinson in the article of Bishop in [3]). Theory of dynamical systems, in particular, the sensitive dependence on initial conditions, the probabilistic description of the behavior of a physical system, and the connections with statistical mechanics pose a host of new problems related to determinism.

Classical mechanics and experiment; connections with other branches of science

Since mechanics is a natural science, the experiment is the ultimate test for correctness of its basic principles. One of its early triumphs is the prediction of the existence and location of the planet Neptune by John Adams (1819-1892) and Urbain Leverier (1811-1877) from the observed irregularities in the motion of Uranus caused by Neptune. Classical mechanics is the theory behind the functioning of any mechanical device, from a bicycle to an airplane.

A recent spectacular success of theory of classical mechanics was the rescue of a Japanese lunar mission in 1991. The spacecraft Hiten was orbiting the Earth with only about 10% of the fuel necessary for it to go to an orbit around the Moon in the "standard" way. Using the subtle gravitational interactions between the Earth and the Moon, Edward Belbruno (b. 1951) succeeded in changing the trajectory of Hiten to send it to an orbit around the Moon, thus salvaging the mission.

Classical mechanics is the oldest area of physics, and its ideas and methods have influenced deeply all other areas of modern physics. The variational principles and the ideas of symmetries and conservation laws developed for the needs of classical mechanics but have been used extensively in all areas of modern physics, and have often been the main guiding principles in deriving new equations or rethinking the old ones. The equations of mechanics of Hamilton and Hamilton-Jacobi were used in developing the quantum mechanics. The principle of relativity of Galileo was one of the main ideas in theory of relativity.

Many developments in mathematics have been motivated by the needs of mechanics. The classical example is Newton's invention of calculus. Topology (originally named “analysis situs”) is a large branch of modern mathematics that was first developed by Poincaré in his work on celestial mechanics. Modern theory of differential equations was revitalized in the 20th century thanks to the problems posed by the theory of dynamical systems. Modern mechanics is also intimately related to geometry, group theory, and other mathematical disciplines. As mentioned above, theory of dynamical systems provides connections with statistical mechanics, ergodic theory, information theory, dimension theory, etc.

Theory of dynamical systems expanded the goals of classical mechanics in new directions. While, before, the main problem was to solve the equations governing the temporal evolution of the system for any particular choice of initial conditions, theory of dynamical systems tries to understand the mechanism responsible for the behavior of the physical system, i.e., to understand *why* the system is behaving in a certain way. To this end, theory of dynamical systems often employs simple *models* that have similar behavior as a complicated system, and analyzes in detail the model—the geometric models for the Lorenz equations mentioned above are an example of this approach. Another such example is the suggestion of the biologist Robert May (b. 1936) in mid-1970s to use simple equations to model complicated ecological phenomena since, despite their simplicity, the solutions of these equations can exhibit a complicated behavior. Around the same time, Mitchell Feigenbaum (b. 1944) studied numerically the so-called *period-doubling bifurcations* of one-dimensional functions, and noticed that certain quantities are *universal*, i.e., that functions can be divided into classes such that these quantities have the same value. Feigenbaum proposed an ingenious explanation of his observations that was based on *renormalization*—an idea borrowed from the arsenal of statistical physics. Namely, he interpreted his numerical results as a manifestation of some phenomena that occur in the *space of functions*—the space where the functions “live”—which is infinitely-dimensional! This suggestion spurred an unprecedented surge of activity and has been employed—with appropriate modifications—to many complicated physical phenomena, deepening our understanding of Nature.

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