Example of computing the derivative of a map

$\vec{f} : \mathbb{R}^2 \to \mathbb{R}^3$, we can endow $\mathbb{R}^2$ and $\mathbb{R}^3$ with any norm (because on finite-dimensional spaces all norms are equivalent). Note that on $\mathbb{R}^n$ we have (check!)

$$|h_j| \le \|\underline{h}\|_p \quad \text{or} \quad \|\underline{h}\|_\infty, \quad p \in [1, \infty).$$

Let

$$\vec{f}(\underline{x}) = \vec{f}(x_1, x_2) = \begin{bmatrix} 1 + 3x_1 \\ x_1^2 x_2 \\ \dfrac{x_1^2}{1 - x_2} \end{bmatrix} = \begin{bmatrix} f_1(\underline{x}) \\ f_2(\underline{x}) \\ f_3(\underline{x}) \end{bmatrix}$$

$$\vec{f}(\underline{x} + \underline{h}) - \vec{f}(\underline{x}) = \begin{bmatrix} 1 + 3(x_1 + h_1) \\ (x_1 + h_1)^2 (x_2 + h_2) \\ \dfrac{(x_1 + h_1)^2}{1 - (x_2 + h_2)} \end{bmatrix} - \begin{bmatrix} 1 + 3x_1 \\ x_1^2 x_2 \\ \dfrac{x_1^2}{1 - x_2} \end{bmatrix}$$

$$f_1(\underline{x}+\underline{h}) - f_1(\underline{x}) = \cancel{1}+3(\cancel{x_1}+h_1) - (\cancel{1}+\cancel{3x_1})$$

$$= 3h_1$$

$$f_2(\underline{x}+\underline{h}) - f_2(\underline{x}) = (x_1+h_1)^2(x_2+h_2) - x_1^2x_2$$

$$= (x_1^2+2x_1h_1+h_1^2)(x_2+h_2) - x_1^2x_2$$

$$= \cancel{x_1^2x_2} + 2x_1x_2h_1 + h_1^2x_2$$
$$+ x_1^2h_2 + 2x_1h_1h_2 + h_1^2h_2 - \cancel{x_1^2x_2}$$

$$= 2x_1x_2h_1 + x_1^2h_2$$
$$+ h_1^2x_2 + 2x_1h_1h_2 + h_1^2h_2$$

$$= 2x_1x_2h_1 + x_1^2h_2$$
$$+ \|\underline{h}\| \underbrace{\left(\frac{h_1}{\|\underline{h}\|}h_1x_2 + 2x_1\frac{h_1}{\|\underline{h}\|}h_2 + \frac{h_2}{\|\underline{h}\|}h_1^2\right)}$$

denote this by $u_2(\underline{h})$;
clearly, $u_2(\underline{h}) \longrightarrow 0$ as $\underline{h} \to \underline{0}$
because $\frac{|h_1|}{\|\underline{h}\|} \le 1$, $\frac{|h_2|}{\|\underline{h}\|} \le 1$.

$$f_3(\underline{x}+\underline{h}) - f_3(\underline{x}) = \frac{(x_1+h_1)^2}{1-(x_2+h_2)} - \frac{x_1^2}{1-x_2}$$

$$= \frac{x_1^2\left(1+\frac{h_1}{x_1}\right)^2}{(1-x_2)\left(1-\frac{h_2}{1-x_2}\right)} - \frac{x_1^2}{1-x_2}$$

$$= \frac{x_1^2}{1-x_2}\left[\left(1+\frac{h_1}{x_1}\right)^2 \sum_{k=0}^{\infty}\left(\frac{h_2}{1-x_2}\right)^k - 1\right]$$

$$= \frac{x_1^2}{1-x_2}\left[\left(1+2\frac{h_1}{x_1}+\frac{h_1^2}{x_1^2}\right)\left(1+\frac{h_2}{1-x_2}+\frac{h_2^2}{(1-x_2)^2}+\cdots\right) - 1\right]$$

$$= \frac{x_1^2}{1-x_2}\Bigg(\cancel{1} + \underline{2\frac{h_1}{x_1}} + \frac{h_1^2}{x_1^2} + \underline{\frac{h_2}{1-x_2}} + 2\frac{h_1}{x_1}\frac{h_2}{1-x_2}$$

$$+ 2\frac{h_1}{x_1}\frac{h_2^2}{(1-x_2)^2} + \frac{h_1^2}{x_1^2} + \frac{h_1^2}{x_1^2}\frac{h_2}{1-x_2}$$

$$+ \frac{h_1^2}{x_1^2}\frac{h_2^2}{(1-x_2)^2} + \cdots - \cancel{1}\Bigg)$$

$$= \frac{2x_1}{1-x_2}h_1 + \frac{x_1^2}{(1-x_2)^2}h_2 + \|\underline{h}\|\, u_3(\underline{h}),$$

where

$$u_3(\underline{h}) = \frac{1}{\|\underline{h}\|}\left[\frac{h_1^2}{x_1^2} + \frac{2h_1h_2}{x_1(1-x_2)} + \frac{2h_1h_2^2}{x_1(1-x_2)^2}\right.$$

$$\left. + \frac{h_1^2}{x_2^2} + \frac{h_1^2h_2}{x_1^2(1-x^2)} + \frac{h_1^2h_2^2}{x_1^2(1-x_2)^2} + \dots\right]$$

and it is clear that

$$u_3(\underline{h}) \longrightarrow 0 \quad \text{as } \underline{h} \to \underline{0}.$$

<u>Putting everything together</u>:

define $u_1(\underline{h}) := 0$,

$$\vec{u} := \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3,$$

$$\vec{u}(\underline{x}) := \begin{bmatrix} u_1(\underline{x}) \\ u_2(\underline{x}) \\ u_3(\underline{x}) \end{bmatrix}$$ where $u_j(\underline{h})$ are

the expressions obtained above;

since $u_j(\underline{h}) \to 0$ as $\underline{h} \to \underline{0}$ $\forall j$,

we have

$$\vec{u}(\underline{h}) \longrightarrow \vec{0} \quad \text{as } \underline{h} \to \underline{0}.$$

$$\vec{f}(\underline{x}+\underline{h})-\vec{f}(\underline{x}) = \begin{bmatrix} 3h_1 + \|\underline{h}\|\, u_1(\underline{h}) \\ 2x_1x_2h_1 + x_1^2 h_2 + \|\underline{h}\|\, u_2(\underline{h}) \\ \dfrac{2x_1}{1-x_2} h_1 + \dfrac{x_1^2}{(1-x_2)^2} h_2 + \|\underline{h}\|\, u_3(\underline{h}) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3 & 0 \\ 2x_1x_2 & x_1^2 \\ \dfrac{2x_1}{1-x_2} & \dfrac{x_1^2}{(1-x_2)^2} \end{bmatrix}}_{} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \|\underline{h}\|\, \vec{u}(\underline{h})$$

this is the derivative $D\vec{f}(\underline{x})$, which can be considered as a linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$D\vec{f}(\underline{x}) = \begin{bmatrix} 3 & 0 \\ 2x_1x_2 & x_1^2 \\ \dfrac{2x_1}{1-x_2} & \dfrac{x_1^2}{(1-x_2)^2} \end{bmatrix} \in L(\mathbb{R}^2, \mathbb{R}^3).$$

Note that the $(\alpha, i)^{\text{th}}$ matrix element of $D\vec{f}(\underline{x})$ is $D\vec{f}(\underline{x})_{\alpha i} = \dfrac{\partial f_\alpha}{\partial x_i}(\underline{x})$.

More about this later ...