Basic Concepts in Differential Geometry

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1 Introduction

Differential geometry concerns the application of linear algebra and calculus to geometric objects. The familiar concepts of distances, lines, planes, and surfaces, when treated in the framework of this field, can be reformulated and simplified to yield numerous applications within mathematics. In particular, this paper will focus on *Riemannian geometry*, the study of real, smooth manifolds equipped with a metric tensor. Like the inner product of linear algebra, the metric tensor is an additional structure which allows one to define distances and angles (and by extension, surface areas and volumes) on a manifold. We will cover a broad range of topics and techniques, from differential forms to the the exterior product to the musical isomorphisms and beyond. This paper aims to explore these concepts in differential geometry I learned with Dr. Petrov, and complement them with examples and applications to demonstrate my familiarity with them. Throughout, we will see many specific instances where the language of differential geometry allows us to condense and generalize familiar ideas in other mathematical fields.

2 Preliminaries

2.1 Definitions/Notations

This is an oversimplified definition, but for our purposes, a *manifold* is a set which locally looks like Euclidean space. The simplest and most important manifold is $\mathbb{R}^n$, and it will be the only one used in this paper (primarily $\mathbb{R}^2$ and $\mathbb{R}^3$ to keep things concise). I will also consider only *smooth* manifolds, in which each neighborhood locally resembling $\mathbb{R}^n$ can be connected to adjacent neighborhoods by C^∞ (infinitely differentiable, or "smooth") functions. Since $\mathbb{R}^n$ is itself a smooth manifold, this point need not be emphasized.

There are a few notations I use here that the reader should be aware of. For one, indices of vector components in vector spaces will be superscripts, whereas indices of *co*vector components in *dual* spaces will be subscripts. Basis coordinates are the opposite in their respective spaces. For example, $v^1\mathbf{e}_1 + v^2\mathbf{e}_2 \in V$ has components (or "coordinate vectors") v^1, v^2 with basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\alpha_1 e^1 + \alpha_2 e^2 \in V^*$ has components α_1, α_2 with basis covectors e^1, e^2. Notice that vectors are in **boldface**. The reason for this is so one can keep track of where each term belongs. It is also foolproof by design; for terms with repeated indices, one must be a subscript and the other must be a superscript in order for a summation to be carried out.

Speaking of which, I will also adopt the Einstein summation convention, in which I suppress the summation symbol (Σ) from terms with repeated indices. For instance,

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + \ldots + v^n\mathbf{e}_n = \sum_{i=1}^{n} v^i\mathbf{e}_i \equiv v^i\mathbf{e}_i.$$

(The number of terms summed over is to be understood from context.) This notation is more compact and efficient, and is quite popular in fields dealing with tensors, such as in relativity.

2.2 Dual Spaces

Many areas in differential geometry deal with vector spaces and their dual spaces, so a brief discussion is warranted. Recall from linear algebra that the *dual space* V^* of a real linear space V is the set of all linear functionals (or *covectors*) on V:

$$V^* := \{\alpha : V \longrightarrow \mathbb{R}\}.$$

Given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for an n-dimensional vector space V, we can construct a basis $\{e^1, e^2, \ldots, e^n\}$ for its dual space V^*, defined by

$$e^i(\mathbf{e}_j) = \delta^i_j, \quad i = 1, \ldots, n.$$

That is, e^i acting on any vector $\mathbf{v} \in V$ picks out its ith component. Notice that $\dim(V) = \dim(V^*)$. There exists a bilinear map $\langle \cdot, \cdot \rangle : V^* \times V \longrightarrow \mathbb{R}$, defined by $\langle \alpha, \mathbf{v} \rangle = \alpha(\mathbf{v})$, called the "natural pairing" between a vector space and its dual. For any covector $\alpha = \alpha_i e^i \in V^*$ and any vector $\mathbf{v} = v^j\mathbf{e}_j \in V$,

$$\langle \alpha, \mathbf{v} \rangle = \langle \alpha_i e^i, v^j\mathbf{e}_j \rangle = \alpha_i v^j \langle e^i, \mathbf{e}_j \rangle = \alpha_i v^j \delta^i_j = \alpha_i v^i \in \mathbb{R},$$

which is the sum of component-wise products.

3 Tangent Spaces

Consider the manifold $\mathcal{M} = \mathbb{R}^m$. For any $\epsilon > 0$, let

$$\boldsymbol{\gamma} := \left(\gamma^1(t), \gamma^2(t), \ldots, \gamma^m(t)\right) : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^m$$

be a differentiable parameterized curve in $\mathcal{M}$ and define $\boldsymbol{\gamma}(0) := \mathbf{x} = (x^1, x^2, \ldots, x^m) \in \mathcal{M}$. The tangent vector to $\boldsymbol{\gamma}$ at $\mathbf{x}$ is then

$$\mathbf{X}_\mathbf{x} := \left(X^1_\mathbf{x}, X^2_\mathbf{x}, \ldots, X^m_\mathbf{x}\right) = \frac{d\boldsymbol{\gamma}}{dt}(0),$$

and is defined by its action on smooth (C^∞) functions. Thus, $\mathbf{X}_\mathbf{x}$ is in fact a map:

$$\mathbf{X}_\mathbf{x} : C^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$$

defined by

$$\mathbf{X}_\mathbf{x}(f) := \frac{d}{dt}\left(f \circ \boldsymbol{\gamma}(t)\right)|_{t=0},$$

and so, by the chain rule,

$$\mathbf{X}_{\mathbf{x}}(f) = \frac{d}{dt}\left(f \circ \boldsymbol{\gamma}(t)\right)|_{t=0} = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i}\left(\boldsymbol{\gamma}(0)\right)\frac{d\gamma^i}{dt}(0) = \frac{d\gamma^i}{dt}(0)\left(\frac{\partial f}{\partial x^i}\right)_{\mathbf{x}}.$$

Finally, dropping the test function f:

$$\mathbf{X}_{\mathbf{x}} = \frac{d\gamma^i}{dt}(0)\left(\frac{\partial}{\partial x^i}\right)_{\mathbf{x}}.$$

In words, the tangent vector $\mathbf{X}_{\mathbf{x}}$ may be represented by a vector sum of components with magnitude $\frac{d\gamma^i}{dt}(0)$ and direction $\frac{\partial}{\partial x^i}$, where the subscripted $\mathbf{x}$ represents the point $\mathbf{x} = \boldsymbol{\gamma}(0) \in \mathcal{M}$ at which the vector $\frac{\partial}{\partial x^i}$ is planted. Notice that we have generalized the notion of a tangent line to a curve in any dimension.

We can generalize even further. For any point $\mathbf{m}$ in $\mathcal{M} = \mathbb{R}^3$, we can endow it with a real linear (vector) space such that for any two vectors $\mathbf{X}_{\mathbf{m}}$ and $\mathbf{Y}_{\mathbf{m}}$ planted at $\mathbf{m}$, any smooth function f on $\mathcal{M}$, and any real number a,

$$\left(\mathbf{X}_{\mathbf{m}} + a\mathbf{Y}_{\mathbf{m}}\right)(f) := \mathbf{X}_{\mathbf{m}}(f) + a\mathbf{Y}_{\mathbf{m}}(f).$$

This linear space, containing *all* tangent vectors to the point $\mathbf{m}$ on the manifold $\mathcal{M}$, is called the *tangent space* to $\mathcal{M}$ at $\mathbf{m}$ and is denoted by $T_{\mathbf{m}}\mathcal{M}$. (The tangent vector $\mathbf{X}_{\mathbf{x}}$ from earlier is just one possible tangent vector at that point, but different curves going through $\mathbf{x}$ will yield different tangent vectors via their directional derivative at that point.)

Furthermore, the tangent spaces at *every* point in a manifold $\mathcal{M}$ can be assembled into a so-called *tangent bundle*, which is itself a manifold. The tangent bundle of $\mathcal{M}$ is defined to be the disjoint union of all tangent spaces of $\mathcal{M}$:

$$T\mathcal{M} := \bigcup_{\mathbf{m}\in\mathcal{M}} T_{\mathbf{m}}\mathcal{M}.$$

To clarify, a tangent bundle is a collection of tangent spaces, and a tangent space is a collection of tangent vectors. It should also be emphasized that tangent vectors and tangent spaces are defined for a specific *point* on $\mathcal{M}$. There exists a special mapping called the *canonical projection*, defined by $\pi : T\mathcal{M} \longrightarrow \mathcal{M}$, which takes in a vector $\mathbf{X}_{\mathbf{m}} \in T_{\mathbf{m}}\mathcal{M} \in T\mathcal{M}$ and retrieves the base point $\mathbf{m}$ at which the tangent space $T_{\mathbf{m}}\mathcal{M}$ is planted. Notice how unlike linear algebra where vectors are traditionally planted at the origin, differential geometry allows us to easily define vector spaces at *any* point in space.

One way to assign a tangent vector to points on a manifold $\mathcal{M}$ is by taking the derivative of a curve $\boldsymbol{\gamma}$ on $\mathcal{M}$, like at the beginning of this section. One can also assign a tangent vector to *every* point on $\mathcal{M}$ by applying a *vector field* to $\mathcal{M}$. Formally, a vector field on a manifold $\mathcal{M}$ is a function $\mathbf{X} : \mathcal{M} \longrightarrow T\mathcal{M}$ such that $\pi \circ \mathbf{X} = \mathrm{id}_{\mathcal{M}}$, the identity mapping in $\mathcal{M}$. The set of all vector fields on $\mathcal{M}$ is denoted by $\mathfrak{X}(\mathcal{M})$. We can endow $\mathfrak{X}(\mathcal{M})$ with a linear space structure: for any $a \in \mathbb{R}$, the vector field $a\mathbf{X} + \mathbf{Y} \in \mathfrak{X}(\mathcal{M})$ at $\mathbf{m} \in \mathcal{M}$ is defined by

$$\left(a\mathbf{X} + \mathbf{Y}\right)_{\mathbf{m}} := a\mathbf{X}_{\mathbf{m}} + \mathbf{Y}_{\mathbf{m}} \in T_{\mathbf{m}}\mathcal{M}.$$

If our manifold is $\mathbb{R}^n$ with Cartesian coordinates, then any tangent vector $\mathbf{X_m} \in T_\mathbf{m}\mathbb{R}^n$ can be expressed in terms of the unit vectors

$$\left(\frac{\partial}{\partial x^1}\right)_\mathbf{m}, \left(\frac{\partial}{\partial x^2}\right)_\mathbf{m}, \ldots, \left(\frac{\partial}{\partial x^n}\right)_\mathbf{m},$$

which in fact form an orthonormal basis in $T_\mathbf{m}\mathbb{R}^n$. In this basis, every tangent vector applied via a vector field $\mathbf{X}$ is automatically endowed with coordinates.

4 Differential Forms

The formal definition of a differential form is difficult to write with what we know so far, but for our purposes, *differential forms* are a generalization of the differential terms like "dx" from calculus. The simplest forms are forms of order 0, called "zero-forms," which are nothing but smooth functions on a manifold $\mathcal{M}$:

$$\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M}) = \{f : \mathcal{M} \longrightarrow \mathbb{R} : f \text{ smooth}\},$$

where $\Omega^0(\mathcal{M})$ is the set of all zero-forms on $\mathcal{M}$.

On the other hand, "one-forms" are simply covectors (i.e., the dual-space equivalent of vectors). The general form of a one-form planted at $\mathbf{m} \in \mathcal{M}$ is $\alpha_\mathbf{m} = \alpha_{\mathbf{m},i} \left(dx^i\right)_\mathbf{m}$, and they live in the "cotangent space" $T^*_\mathbf{m}\mathcal{M}$ dual to the tangent space. Here, $\left(dx^i\right)_\mathbf{m} \in T^*_\mathbf{m}\mathcal{M}$ is a linear functional $\left(dx^i\right)_\mathbf{m} : T_\mathbf{m}\mathcal{M} \to \mathbb{R}$ which acts on tangent vectors via the natural pairing:

$$\left\langle \left(dx^i\right)_\mathbf{m}, \left(\frac{\partial}{\partial x^j}\right)_\mathbf{m} \right\rangle = \delta^i_j.$$

We need additional tools to define differential forms of higher order. One such tool is the *tensor product*, denoted $V^* \otimes V^*$, and is defined as the set of all bilinear mappings $V \times V \longrightarrow \mathbb{R}$. Elements of $V^* \otimes V^*$ are $\alpha \otimes \beta = \alpha_i \beta_j \, e^i \otimes e^j$, and a basis for $V^* \otimes V^*$ is all combinations of $e^i \otimes e^j$, so $\dim\left(V^* \otimes V^*\right) = \dim(V)^2$. Another tool we need is the *wedge product* (or *exterior product*), defined in terms of the tensor product:

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha.$$

The wedge product is alternating by design, meaning if σ is a permutation of $1, 2, \ldots, k$, then $\alpha_{\sigma(1)} \wedge \alpha_{\sigma(2)} \wedge \ldots \wedge \alpha_{\sigma(k)} = \operatorname{sgn}(\sigma)\, \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k$. Specifically, $\alpha \wedge \beta = -\beta \wedge \alpha$ and $\alpha_i \wedge \alpha_i = 0$ for any index i.

With the concept of wedge products in mind, we can define higher-order forms. A two-form, for instance, is of the form

$$\omega = \sum_{i<j} \omega_{ij} \; dx^i \wedge dx^j \in \Omega^2(\mathcal{M})$$

and it acts on *two* vectors, like this:

$$\omega_\mathbf{m}\left(\mathbf{X_m}, \mathbf{Y_m}\right) = \sum_{i<j} \omega_{\mathbf{m},ij} \left(X^i_\mathbf{m} Y^j_\mathbf{m} - X^j_\mathbf{m} Y^i_\mathbf{m}\right).$$

Finally, we can express a general k-form as follows:

$$\varphi = \sum_{i_1<\cdots<i_k} \varphi_{i_1\cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(\mathcal{M}),$$

where the coefficients $\varphi_{i_1\cdots i_k}$ are real numbers.

As an example, let's consider all possible forms on $\mathbb{R}^3$, using the dual basis vectors e^1, e^2, and e^3. There is only one three-form ($e^1 \wedge e^2 \wedge e^3$; barring any permuations), there are three each of two-forms ($e^1 \wedge e^2$, $e^2 \wedge e^3$, $e^3 \wedge e^1$) and one-forms (e^1, e^2, e^3), and there is one zero-form (just an arbitrary smooth function, f). Similarly, in $\mathbb{R}^2$, there is only one two-form ($e^1 \wedge e^2$), two one-forms (e^1, e^2), and one arbitrary smooth function for a zero-form.

There exists another operation called the *exterior derivative*, which takes a k-form and produces a $(k+1)$-form:

$$d : \Omega^k(\mathcal{M}) \longrightarrow \Omega^{k+1}(\mathcal{M}).$$

To find the exterior derivative df of a zero-form (i.e., a smooth function) f, define

$$\langle (df)_{\mathbf{m}}, \mathbf{X}_{\mathbf{m}} \rangle := \mathbf{X}_{\mathbf{m}}(f) = X^i_{\mathbf{m}} \left(\frac{\partial f}{\partial x^i} \right)_{\mathbf{m}},$$

and comparing to the earlier definition of the natural pairing, we get

$$df = \frac{\partial f}{\partial x^i}\, dx^i.$$

For a $(k \geq 1)$-form, its exterior derivative is defined differently. If

$$\varphi = \sum_{i_1<\cdots<i_k} \varphi_{i_1\cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

is a general k-form, then

$$d\varphi := \left(\sum_{\ell} \frac{\partial \varphi_{i_1\cdots i_k}}{\partial x^\ell}\, dx^\ell \right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The exterior derivative is a key instrument needed for writing the generalized Stokes theorem, which we'll see later. However, we must discuss integration, as well as differentiation, if we are to accomplish this.

5 Integration of Forms

Differential k-forms may be integrated over k-dimensional manifolds. Let's start with the $k = 1$ case, which should be very reminiscent of introductory calculus. Let the one-dimensional manifold be a curve Γ in an n-dimensional manifold $\mathcal{M}$ with parameterization $\boldsymbol{\gamma} : [a, b] \longrightarrow \mathcal{M}$, and let α be a one-form on $\mathcal{M}$. Then, $\frac{d\boldsymbol{\gamma}(t)}{dt} \in T_{\boldsymbol{\gamma}(t)}\mathcal{M}$ and $\alpha_{\boldsymbol{\gamma}(t)} \in T^*_{\boldsymbol{\gamma}(t)}\mathcal{M}$. The integral of α over the curve Γ is defined to be

$$\int_\Gamma \alpha := \int_a^b \left\langle \alpha_{\boldsymbol{\gamma}(t)}, \left(\frac{d\boldsymbol{\gamma}(t)}{dt} \right)_{\boldsymbol{\gamma}(t)} \right\rangle dt = \int_a^b \alpha_i \frac{d\gamma^i(t)}{dt} dt,$$

where $\langle \cdot, \cdot \rangle : T^*\mathcal{M} \times T\mathcal{M} \longrightarrow \mathbb{R}$ is the natural pairing, and the subscript coordinates in the right-most expression have been removed for clarity.

Let's consider an example to see how exactly this ties back to calculus. Let $\alpha = 5x\,dx + xy\,dy \in T^*\mathbb{R}^2$ be a one-form on $\mathbb{R}^2$, and let the curve of integration be $y = x^2$ from $x = 0$ to $x = 2$. (Note that we will still integrate over $\mathbb{R}$ since the curve can be parameterized by a single variable; the one-form being defined over $\mathbb{R}^2$ simply means the curve lies entirely in the Euclidean plane.) Parameterize the curve by $\boldsymbol{\gamma} : [0, 2] \longrightarrow \mathbb{R}^2 : t \longmapsto \boldsymbol{\gamma}(t) := (t, t^2)$. Then, $\frac{d\boldsymbol{\gamma}(t)}{dt} = (1, 2t) = \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y} \in T\mathbb{R}^2$. We can rewrite α by replacing x with t and y with t^2 to obtain $\alpha = 5t\,dx + t^3\,dy$. Now, everything is in the appropriate form to carry out the integration:

$$\int_\Gamma \alpha = \int_0^2 \left\langle 5t\,dx + t^3\,dy, \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y} \right\rangle dt = \int_0^2 5t(1) + t^3(2t)\,dt = \frac{114}{5}.$$

You may notice that the calculations used to perform this integration are strikingly similar to a formula seen in vector calculus, namely,

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt.$$

We did perform a sort of dot product when we added pairwise products between basis vectors, and this is no coincidence; these formulas are actually the same thing written in different ways. Therefore, we come to an important realization: The integration of one-forms along a one-dimensional manifold (i.e., a curve) is a generalization of the *line integral* from vector calculus.

In higher dimensions, the same relationships occur. Two-forms integrated along two-dimensional manifolds (i.e., surfaces) are generalizations of *surface integrals*, and three-forms integrated along three-dimensional manifolds (i.e., volumes) are generalizations of *volume integrals*. For an n-form $\omega \in \Omega^n(\mathcal{M})$ integrated over an n-dimensional manifold $\mathcal{M}$, the integration is defined such that the wedge product becomes a usual product:

$$\underbrace{\int \cdots \int_{\mathcal{M}}}_{n \text{ times}} \omega = \int \cdots \int_{\mathcal{M}} \omega_{1\cdots n}(x^1, \ldots, x^n)\,dx^1 \wedge \cdots \wedge dx^n := \int \cdots \int_{\mathcal{M}} \omega_{1\cdots n}(x^1, \ldots, x^n)\,dx^1 \cdots dx^n.$$

6 Volume Forms

6.1 The Hodge Star Operator

If V is an n-dimensional inner product space, then we can define the so-called "p-fold exterior product" of V by

$$\Lambda^p V := \underbrace{V \wedge V \wedge \cdots \wedge V}_{p \text{ times}} = \{\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_p : \mathbf{v}_i \in V, i = 1, 2, \ldots, p\},$$

where $p \leq n$. We call elements of $\Lambda^p V$ "p-vectors." We can obtain an induced inner product $\langle \cdot, \cdot \rangle_\Lambda : \Lambda^p V \times \Lambda^p V \longrightarrow \mathbb{R}$ on $\Lambda^p V$ using the inner product $\langle \cdot, \cdot \rangle$ endowed to V. Given two elements

$\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p$ and $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_p$ in $\Lambda^p V$, we define

$$\langle \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p, \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_p \rangle_\Lambda := \det\left(\langle \mathbf{v}_i, \mathbf{w}_j \rangle\right) = \begin{vmatrix} \langle \mathbf{v}_1, \mathbf{w}_1 \rangle & \langle \mathbf{v}_1, \mathbf{w}_2 \rangle & \cdots & \langle \mathbf{v}_1, \mathbf{w}_p \rangle \\ \langle \mathbf{v}_2, \mathbf{w}_1 \rangle & \langle \mathbf{v}_2, \mathbf{w}_2 \rangle & \cdots & \langle \mathbf{v}_2, \mathbf{w}_p \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{v}_p, \mathbf{w}_1 \rangle & \langle \mathbf{v}_p, \mathbf{w}_2 \rangle & \cdots & \langle \mathbf{v}_p, \mathbf{w}_p \rangle \end{vmatrix},$$

which is simply the Gram determinant from linear algebra. Furthermore, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is an orthonormal basis for V, then

$$\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_p} \quad \text{with} \quad 1 \leq i_1 < i_2 < \cdots < i_p \leq n$$

forms an orthonormal basis for $\Lambda^p V$. By convention, if we have a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ for V, we can declare that this basis has a positive orientation. Any other basis for V obtained from this one via a change-of-basis matrix with *positive* determinant is also positive, and likewise if the change-of-basis matrix has *negative* determinant, that basis is negative. Note that whichever basis we initially chose to be positive is arbitrary; the key thing here is distinguishing bases by their orientation relative to one another.

If V is an n-dimensional inner product space with a given orientation, we can define the so-called "Hodge star operator" $*$ on its exterior product:

$$* : \Lambda^p V \longrightarrow \Lambda^{n-p} V \quad \text{where} \quad * \left(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_p}\right) := \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{n-p}}.$$

Here, the indices $j_1, \ldots, j_{n-p}$ are chosen such that $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_p}, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-p}}$ is a positive basis for V, and $0 \leq p \leq n$ (a 0-vector is merely a number, or *scalar*). If it were a negative basis, then the only difference is the presence of a minus sign. By definition,

$$*(1) = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \quad \text{and} \quad * \left(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n\right) = 1.$$

So, the Hodge star operator takes a p-vector and yields an $(n-p)$-vector. To illustrate what this operator does more clearly, let's consider the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbb{R}^3$. The following are true:

$$*(\mathbf{i}) = \mathbf{j} \wedge \mathbf{k}, \qquad *(\mathbf{j}) = \mathbf{k} \wedge \mathbf{i}, \qquad *(\mathbf{k}) = \mathbf{i} \wedge \mathbf{j},$$

$$*(\mathbf{i} \wedge \mathbf{j}) = \mathbf{k}, \qquad *(\mathbf{j} \wedge \mathbf{k}) = \mathbf{i}, \qquad *(\mathbf{k} \wedge \mathbf{i}) = \mathbf{j},$$

$$*(\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}) = 1, \qquad *(1) = \mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}.$$

In $\mathbb{R}^3$, the 2-vectors (also known as *bivectors*) $\mathbf{j} \wedge \mathbf{k}, \mathbf{k} \wedge \mathbf{i}, \mathbf{i} \wedge \mathbf{j} \in \Lambda^2(\mathbb{R}^3)$ represent oriented parallelograms spanned by their component vectors. The Hodge star operator acts on these bivectors to produce a vector perpendicular to their plane. For example, $\mathbf{i} \wedge \mathbf{j}$ represents a parallelogram in the xy-plane, and $\mathbf{k} = *(\mathbf{i} \wedge \mathbf{j})$ is the unit vector along the z-axis, which is clearly perpendicular to the xy-plane.

One interesting application of $*$ in $\mathbb{R}^3$ is that it directly relates to the cross product. Let $\mathbf{u} = (u^1, u^2, u^3)$ and $\mathbf{v} = (v^1, v^2, v^3)$ be two vectors in $\mathbb{R}^3$. Then,

$$\begin{aligned} \mathbf{u} \wedge \mathbf{v} &= (u^1\mathbf{i} + u^2\mathbf{j} + u^3\mathbf{k}) \wedge (v^1\mathbf{i} + v^2\mathbf{j} + v^3\mathbf{k}) \\ &= (u^2v^3 - u^3v^2)\mathbf{j} \wedge \mathbf{k} + (u^3v^1 - u^1v^3)\mathbf{k} \wedge \mathbf{i} + (u^1v^2 - u^2v^1)\mathbf{i} \wedge \mathbf{j}, \end{aligned}$$

and so,

$$\begin{aligned}
*(\mathbf{u} \wedge \mathbf{v}) &= (u^2v^3 - u^3v^2) * (\mathbf{j} \wedge \mathbf{k}) + (u^3v^1 - u^1v^3) * (\mathbf{k} \wedge \mathbf{i}) + (u^1v^2 - u^2v^1) * (\mathbf{i} \wedge \mathbf{j}) \\
&= (u^2v^3 - u^3v^2)\mathbf{i} + (u^3v^1 - u^1v^3)\mathbf{j} + (u^1v^2 - u^2v^1)\mathbf{k} \\
&= \mathbf{u} \times \mathbf{v}.
\end{aligned}$$

As we will see later, the Hodge star operator has other applications to vector calculus as well.

6.2 The Metric Tensor

Let's now introduce another important concept: the *metric tensor* (or simply *metric*), one of the most fundamental quantities in differential geometry. Recall that if $\mathcal{M}$ is our manifold, then at every point $\mathbf{m} \in \mathcal{M}$ there is a tangent vector space $T_{\mathbf{m}}\mathcal{M}$ consisting of all vectors tangent to $\mathcal{M}$ at $\mathbf{m}$. The metric tensor at $\mathbf{m}$ is a map $g_{\mathbf{m}} : T_{\mathbf{m}}\mathcal{M} \times T_{\mathbf{m}}\mathcal{M} \longrightarrow \mathbb{R}$ which is (i) symmetric, so that $g(\mathbf{u}, \mathbf{v}) = g(\mathbf{v}, \mathbf{u})$, (ii) positive definite, so that $g(\mathbf{u}, \mathbf{u}) \geq 0$, and (iii) bilinear, so that $g(a\mathbf{u}_1 + \mathbf{u}_2, b\mathbf{v}_1 + \mathbf{v}_2) = ag(\mathbf{u}_1) + g(\mathbf{u}_2) + bg(\mathbf{v}_1) + g(\mathbf{v}_2)$. A manifold $\mathcal{M}$ together with a metric tensor g is called a *Riemannian manifold* and is denoted by $(\mathcal{M}, g)$. If $\mathbf{X_m}$ and $\mathbf{Y_m}$ are two tangent vectors at $\mathbf{m}$, then

$$g_{\mathbf{m}}(\mathbf{X_m}, \mathbf{Y_m}) = \sum_{i,j} g_{\mathbf{m},ij} X^i_{\mathbf{m}} Y^j_{\mathbf{m}},$$

where the components g_{ij} depend on the coordinate system used.

Because they have two indices (i and j), metrics can be represented by a matrix:

$$g = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}.$$

For example, one of the simplest metric tensors is the two-dimensional "Euclidean metric" in Cartesian coordinates, which is just the 2×2 identity matrix:

$$g_{(x,y)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, $g_{(x,y)} = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$. In general, the Euclidean metric on $\mathbb{R}^n$ is just the $n \times n$ identity matrix $\sum_{i=1}^n dx^i \otimes dx^i$ (again, assuming Cartesian coordinates).

Using this identity matrix as a baseline, we can find the metric tensor in other coordinate systems. If $(x^1, \ldots, x^n)$ are the Cartesian coordinates and $(y^1, \ldots, y^n)$ are the new coordinates, we first express the y coordinates in terms of the original x coordinates. The matrix elements of the new metric tensor are found by:

$$g_{ij} = \frac{\partial(y^1, \ldots, y^n)}{\partial x^i} \cdot \frac{\partial(y^1, \ldots, y^n)}{\partial x^j},$$

where the "$\cdot$" is the Euclidean dot product. For example, what if we used cylindrical polar coordinates instead? To find the metric tensor in this case, we need to apply the coordinate transformations: $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Letting $\mathbf{r} = (x, y) = (\rho\cos\theta, \rho\sin\theta)$, we have

$$\begin{aligned}
g_{11} &= \frac{\partial \mathbf{r}}{\partial \rho} \cdot \frac{\partial \mathbf{r}}{\partial \rho} = (\cos\theta, \sin\theta) \cdot (\cos\theta, \sin\theta) = \sin^2\theta + \cos^2\theta = 1 \\
g_{12} &= \frac{\partial \mathbf{r}}{\partial \rho} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = (\cos\theta, \sin\theta) \cdot (-\rho\sin\theta, \rho\cos\theta) = 0 = g_{21} \\
g_{22} &= \frac{\partial \mathbf{r}}{\partial \theta} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = (-\rho\sin\theta, \rho\cos\theta) \cdot (-\rho\sin\theta, \rho\cos\theta) = \rho^2(\sin^2\theta + \cos^2\theta) = \rho^2.
\end{aligned}$$

Thus,

$$g_{(\rho,\theta)} = \begin{bmatrix} 1 & 0 \\ 0 & \rho^2 \end{bmatrix}.$$

Similarly, one can find that in $\mathbb{R}^3$,

$$g_{(x,y,z)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad g_{(\rho,\theta,z)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad g_{(r,\theta,\phi)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

are the metric tensors in Cartesian, cylindrical polar, and spherical polar coordinates, respectively.

To see exactly how the metric tensor gives rise to geometric concepts, as hinted at earlier, consider the following example. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in the Riemannian manifold $(\mathbb{R}^n, g)$. The angle between them is found in exactly the same way as in linear algebra, but replacing the inner product with $g(\mathbf{u}, \mathbf{v})$:

$$\cos\theta = \frac{g(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|\|\mathbf{v}\|},$$

where $\|\mathbf{w}\| := \sqrt{g(\mathbf{w}, \mathbf{w})}$ is the norm induced by the metric g. Bear in mind that the coordinate system used for the vectors $\mathbf{u}$ and $\mathbf{v}$ must be the same as that used for the metric g. For simplicity, assume Cartesian coordinates. Then, $g = \sum_{i=1}^{n} dx^i \otimes dx^i$ and $g(\mathbf{u}, \mathbf{v}) = \sum_{i,j} g_{ij} u^i v^j = (1)u^1v^1 + \cdots + (1)u^n v^n$, which is exactly the same as the dot product. This also means that the norm/length of $\mathbf{u}$ is $\|\mathbf{u}\| = \sqrt{(u^1)^2 + \cdots + (u^n)^2}$. Thus, we see how a metric tensor can give rise to both angles and lengths.

Once we know how to compute angles between vectors and lengths of vectors, we can develop a rudimentary concept of area. Recall that the magnitude of the cross product $\|\mathbf{u} \times \mathbf{v}\|$ is equal to the area of the parallelogram whose sides have length $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$. Now, we know that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$, so using the definitions of length and angle above, we find that

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{g(\mathbf{u}, \mathbf{u})}\sqrt{g(\mathbf{v}, \mathbf{v})}\sqrt{1 - \left(\frac{g(\mathbf{u}, \mathbf{v})}{\sqrt{g(\mathbf{u}, \mathbf{u})}\sqrt{g(\mathbf{v}, \mathbf{v})}}\right)^2} = \sqrt{g(\mathbf{u}, \mathbf{u})g(\mathbf{v}, \mathbf{v}) - g(\mathbf{u}, \mathbf{v})^2},$$

where we used the fact that $\sin(\arccos x) = \sqrt{1 - x^2}$. So, the area of this parallelogram can be defined solely in terms of the metric tensor. To be able to define areas and volumes in general, we need to discuss an important application of the metric tensor: the volume form.

6.3 The Volume Form

The general definition of a volume form is simply any top-dimensional differential form on a manifold. That is, if $\mathcal{M}$ is an n-dimensional manifold, then a volume form on $\mathcal{M}$ is simply an n-form on $\mathcal{M}$. Natural volume forms exist for *Riemannian* manifolds, however; the "Riemannian volume form" on $(\mathcal{M}, g)$ is given in terms of the metric tensor g by

$$\mathrm{vol}_g := \sqrt{|g_{ij}|}\, dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(\mathcal{M}),$$

where $|g_{ij}|$ is the determinant of the metric tensor. For instance, since the metric tensor in $\mathbb{R}^n$ with Cartesian coordinates is just the $n \times n$ identity matrix, whose determinant is unity, then the volume form is simply $dx^1 \wedge \cdots \wedge dx^n$. Using the metrics above in $\mathbb{R}^3$ for Cartesian, cylindrical polar, and spherical polar coordinates, one can show that the volume forms on $\mathbb{R}^3$ using these metrics are

$$\mathrm{vol}_{g_{(x,y,z)}} = dx \wedge dy \wedge dz, \quad \mathrm{vol}_{g_{(\rho,\theta,z)}} = \rho\, d\rho \wedge d\theta \wedge dz, \quad \mathrm{vol}_{g_{(r,\theta,\phi)}} = r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi,$$

respectively. Notice that any n-form can be written as a multiple of the volume form, since the exterior product of an n-form with a zero-form is still an n-form.

Recall that n-forms can be integrated over n-dimensional manifolds. Sticking with $\mathbb{R}^n$ as our manifold, let $\mathrm{vol}_g = \sqrt{|g_{ij}|}\, dx^1 \wedge \cdots \wedge dx^n$ be the volume form we work with, and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a generic function. The integral of f over $\mathbb{R}^n$ is defined by

$$\int_{\mathbb{R}^n} f \ \mathrm{vol}_g := \int \cdots \int f(x^1, \ldots, x^n) \sqrt{|g_{ij}|}\, dx^1 \cdots dx^n.$$

Thus, we define that the integration of a volume form transforms into a usual integral from multivariable calculus, replacing the exterior product with the normal product of differential terms $dx^1, \ldots, dx^n$. If we take the three coordinate systems in $\mathbb{R}^3$ above, we see that integration of a function f will transform to

$$\iiint f(x,y,z)\, dx\, dy\, dz, \qquad \iiint f(\rho,\theta,z)\, \rho\, d\rho\, d\theta\, dz, \qquad \iiint f(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi,$$

which are standard triple integrals as seen in a multivariable calculus course. An important realization can be made; the volume form simply transforms into the volume element dV, and the $\sqrt{|g_{ij}|}$ term at the front of the volume form transforms into the Jacobian factor in each coordinate system. So, we see that the volume form is a tool which provides a link between integration of differential forms to integration of functions, connecting the worlds of differential geometry and calculus.

Despite the name, volume forms can symbolize more than just a geometric volume via a triple integral. If we work in the manifold $\mathbb{R}$, the volume form is just dx, so our integral is nothing special:

$$\int f \,\mathrm{vol} = \int f(x)\, dx,$$

i.e., the signed area under the curve f. Similarly, working in $\mathbb{R}^2$ yields a double integral, which symbolizes the signed volume under the surface $f(x,y)$:

$$\iint f \,\mathrm{vol} = \iint f(x,y)\, dx\, dy.$$

The integrand depends on which coordinate system we work in, since different coordinate systems have different metric tensors.

Here we have another crucial fact: Since we're defining integration of a function using its volume form, and since volume forms depend on the metric tensor g, we see that any calculation involving integrals of functions can be formulated in terms of the metric tensor. Thus, areas, volumes, arc lengths, surface areas, centers of mass, moments of inertia, and other geometric calculations are *all* attainable via the metric tensor, as we have previously stated. Volume forms were the "missing link" we needed to fully realize the geometric applications of the metric tensor.

7 Stokes' Theorem and Other Applications

7.1 The Musical Isomorphisms and Hypersurfaces

Another way we can use the metric tensor is by forming a direct relationship between the tangent bundle and cotangent bundle of a Riemannian manifold. If $(\mathcal{M}, g)$ is our manifold and $\mathbf{m} \in \mathcal{M}$ is any point in our manifold, then we define the so-called "flat" or "lowering" operator by

$$g^\flat_{\mathbf{m}} : T_{\mathbf{m}}\mathcal{M} \longrightarrow T^*_{\mathbf{m}}\mathcal{M} : \quad \mathbf{X}_{\mathbf{m}} \longmapsto g^\flat_{\mathbf{m}}(\mathbf{X}_{\mathbf{m}}) := \mathbf{X}_{\mathbf{m}} \lrcorner\, g_{\mathbf{m}},$$

where the $\lrcorner$ symbol indicates an operation called the *interior product*, which is defined by $\mathbf{X}_{\mathbf{m}} \lrcorner\, g_{\mathbf{m}} := g_{\mathbf{m}}(\mathbf{X}_{\mathbf{m}}, \cdot) = g_{ij,\mathbf{m}} X^j_{\mathbf{m}}\, (dx^i)_{\mathbf{m}}$. So, we can write

$$g^\flat_{\mathbf{m}}(\mathbf{X}_{\mathbf{m}}) = g_{ij,\mathbf{m}} X^j_{\mathbf{m}}\, (dx^i)_{\mathbf{m}}.$$

One property of metric tensors is that they are nondegenerate, meaning their determinant is nonzero and so they have an inverse. Therefore, we can define an inverse process called the "sharp" or "raising" operator:

$$g^\sharp_{\mathbf{m}} := (g^\flat_{\mathbf{m}})^{-1} : T^*_{\mathbf{m}}\mathcal{M} \longrightarrow T_{\mathbf{m}}\mathcal{M}.$$

We can write this more explicitly using coordinates. Let $(g^{jk}_{\mathbf{m}})$ be the inverse matrix of $(g_{ij,\mathbf{m}})$. By definition, $\sum_j g_{ij,\mathbf{m}} g^{jk}_{\mathbf{m}} = \delta^i_k$. Using that $g^\sharp_{\mathbf{m}} = (g^\flat_{\mathbf{m}})^{-1}$, we set $\mathbf{X}_{\mathbf{m}} = g^\sharp_{\mathbf{m}}(\alpha_{\mathbf{m}})$ for some $\alpha_{\mathbf{m}} \in T_{\mathbf{m}}\mathcal{M}$. From $[g^\flat_{\mathbf{m}}(\mathbf{X}_{\mathbf{m}})]_i = X^j_{\mathbf{m}} g_{ij,\mathbf{m}}$, we get $\alpha_{i,\mathbf{m}} = [g^\sharp_{\mathbf{m}}(\alpha_{\mathbf{m}})]^j g_{ji,\mathbf{m}}$. Now, multiplying both sides by $g^{ik}_{\mathbf{m}}$ and summing over i, we get

$$\alpha_{i,\mathbf{m}} g^{ik}_{\mathbf{m}} = [g^\sharp_{\mathbf{m}}(\alpha_{\mathbf{m}})]^j g_{ji,\mathbf{m}} g^{ik}_{\mathbf{m}} = [g^\sharp_{\mathbf{m}}(\alpha_{\mathbf{m}})]^j \delta^k_j = [g^\sharp_{\mathbf{m}}(\alpha_{\mathbf{m}})]^k.$$

Thus, $[g^\sharp_{\mathbf{m}}(\alpha_{\mathbf{m}})]^i = g^{ij}_{\mathbf{m}} \alpha_{j,\mathbf{m}}$ and so

$$g^\sharp_{\mathbf{m}}(\alpha_{\mathbf{m}}) = g^{ij}_{\mathbf{m}} \alpha_{i,\mathbf{m}} \left(\frac{\partial}{\partial x^j}\right)_{\mathbf{m}}.$$

Similar to how the metric g acts on tangent vectors in the tangent bundle, the *inverse metric*

$$\tilde{g} := \sum_{i,j} g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

acts on covectors (one-forms) from the cotangent bundle. If $\alpha_{\mathbf{m}}$ and $\beta_{\mathbf{m}}$ are two covectors planted at $\mathbf{m} \in \mathcal{M}$, then

$$\tilde{g}(\alpha_{\mathbf{m}}, \beta_{\mathbf{m}}) := \left\langle \alpha_{\mathbf{m}}, g^\flat_{\mathbf{m}}(\beta_{\mathbf{m}}) \right\rangle = \sum_{i,j} g^{i,j}_{\mathbf{m}}\, \alpha_{i,\mathbf{m}} \beta_{j,\mathbf{m}}.$$

Together, the flat and sharp operators define the *musical isomorphisms*, called so because $\flat$ and $\sharp$ represent lowering and raising operators in music as they do in differential geometry. If we ignore the planting point $\mathbf{m}$ and consider these operators as acting on the whole manifold, then these operators represent isomorphisms between the vector fields on $\mathcal{M}$ and the one-forms on $\mathcal{M}$:

$$g^\flat : \mathfrak{X}(\mathcal{M}) \longrightarrow \Omega^1(\mathcal{M}) : \quad g^\flat(\mathbf{X}) = g_{ij} X^i \, dx^j$$

and

$$g^\sharp : \Omega^1(\mathcal{M}) \longrightarrow \mathfrak{X}(\mathcal{M}) : \quad g^\sharp(\alpha) = g^{ij} \alpha_i \, \frac{\partial}{\partial x^j},$$

where again $g^\flat$ and $g^\sharp$ are inverse operators:

$$g^\sharp(g^\flat(\mathbf{X})) = \mathbf{X} \qquad \text{and} \qquad g^\flat(g^\sharp(\alpha)) = \alpha.$$

If $\mathcal{M} = \mathbb{R}^n$ is an n-dimensional manifold, then we call $\Sigma \subset \mathbb{R}^{n-1}$ a *hypersurface*, which has dimension $n-1$ and is embedded in the ambient manifold $\mathbb{R}^n$. We can use the musical isomorphisms, the metric tensor, and the volume form on the manifold together to visualize what the *induced* volume form on a hypersurface might look like.

Consider a sphere of radius R centered at the origin. Now,

$$g = (g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix} \qquad \text{and} \qquad \tilde{g} = (g^{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix}$$

are the metric tensor and inverse metric, respectively, on $\mathbb{R}^3$ using spherical polar coordinates. The volume form, as mentioned earlier, is $\mathrm{vol} = r^2 \sin\theta \; dr \wedge d\theta \wedge d\phi$. The volume form on the hypersurface Σ induced by vol is given by $\mathrm{vol}_\Sigma := (\mathbf{N} \lrcorner \, \mathrm{vol})\,|_\Sigma \in \Omega^{n-1}(\Sigma)$, where "$|_\Sigma$" indicates that $\mathbf{N} \lrcorner \, \mathrm{vol}$ is restricted to the hypersurface Σ, and $\mathbf{N}$ is the unit normal vector pointing outward from Σ and is given by

$$\mathbf{N} := g^\sharp(d\Sigma) = g^{ij} \frac{\partial \Sigma}{\partial x^i} \frac{\partial}{\partial x^j}.$$

The hypersurface in this case is just the surface of the sphere, which is given by $r = R$, so let $\Sigma := r - R = 0$ represent our hypersurface. Since only diagonal terms in the inverse metric exist, we only sum three terms when calculating the normal vector $\mathbf{N}$:

$$\mathbf{N} = g^{11} \frac{\partial \Sigma}{\partial r} \frac{\partial}{\partial r} + g^{22} \frac{\partial \Sigma}{\partial \theta} \frac{\partial}{\partial \theta} + g^{33} \frac{\partial \Sigma}{\partial \phi} \frac{\partial}{\partial \phi} = (1)(1) \frac{\partial}{\partial r} + 0 + 0 = \frac{\partial}{\partial r},$$

which makes sense since the normal vector on a sphere's surface always points radially outward. The other two terms vanish because the surface depends only on r, and notice also that $\mathbf{N}$ is already normalized. Then, we have

$$\begin{aligned} \mathbf{N} \lrcorner \, \mathrm{vol} &= \sum_{i=1}^{3} (-1)^{i-1} N^i \; dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^3 \\ &= (1)(r^2 \sin\theta) \; d\theta \wedge d\phi - (0)(r^2 \sin\theta) \; dr \wedge d\phi + (0)(r^2 \sin\theta) \; dr \wedge d\theta \\ &= r^2 \sin\theta \; d\theta \wedge d\phi, \end{aligned}$$

where the hat notation $\widehat{dx^i}$ means that term is excluded in the sum. When restricted to the surface Σ, we get

$$\mathrm{vol}_\Sigma = (\mathbf{N} \lrcorner \mathrm{vol})\,|_{r=R} = R^2 \sin\theta \; d\theta \wedge d\phi.$$

Notice that whereas the volume form on $\mathbb{R}^3$ is a volume element, the volume form on a two-dimensional hypersurface is an *area* element; specifically, vol_Σ is the differential normal area in spherical coordinates in the direction of $\hat{\mathbf{r}} = \frac{\partial}{\partial r}$. Thus, we see how the raising operator $\sharp$ together with the metric and volume form can be used to contract a volume element of a sphere to an area element on its surface.

7.2 Vector Calculus

For the next application, let's discuss how the operations of vector calculus have equivalent formulations in the language of differential geometry.

We have seen already how the cross product in $\mathbb{R}^3$ can be expressed via the exterior product and Hodge star operator: $\mathbf{u} \times \mathbf{v} = *(\mathbf{u} \wedge \mathbf{v})$, and also how the inner product is defined in terms of the metric tensor: $\langle \mathbf{u}, \mathbf{v} \rangle := g(\mathbf{u}, \mathbf{v}) := g_{ij} u^i v^j$. If f is a smooth function, recall that the *gradient* of f is a vector which describes the magnitude and direction of steepest ascent of f at a given point. The gradient has an equivalent formulation in terms of the sharp operator and the exterior derivative:

$$\nabla f = g^\sharp(df).$$

Let's check that this holds in $\mathbb{R}^3$ with Cartesian coordinates. Using $df = \frac{\partial f}{\partial x^i}\, dx^i$ and that $g^\sharp(\alpha) = g^{ij} \alpha_i \frac{\partial}{\partial x^j}$ for any one-form α, the left-hand side becomes

$$g^{ij}\, df_i \frac{\partial}{\partial x^j} = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

In $\mathbb{R}^3$ with Cartesian coordinates, $g^{ij} = \delta^{ij}$ is the 3×3 identity matrix. Furthermore, we write the coordinates x^1, x^2, x^3 as x, y, z, respectively, and for the unit vectors $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$ we write $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively. The left-hand side can thus be written as

$$\delta^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k},$$

which is exactly the expression for the gradient from vector calculus.

The other vector calculus operations can be expressed in the language of differential geometry as well, but I won't derive them all for the sake of brevity. Instead, I'll focus on the generalized Stokes theorem, which is one of the most important results in vector calculus. If $\mathcal{M}$ is an n-dimensional manifold and $\partial\mathcal{M}$ is its $(n-1)$-dimensional boundary, then the *generalized Stokes theorem* holds that

$$\int_{\mathcal{M}} d\omega = \oint_{\partial\mathcal{M}} \omega,$$

where ω is an $(n-1)$-form and $d\omega$ is its exterior derivative, an n-form. It turns out that this equation generalizes several major theorems from calculus, so let's explore how this happens.

Suppose our manifold is $\mathcal{M} = [a,b] \subset \mathbb{R}$ and $\omega = f(x) \in \Omega^0([a,b])$ is our differential form. Then, $\partial\mathcal{M} = \{a_-, b_+\}$ and $d\omega = \frac{df}{dx}\,dx$, where the subscripts on a_- and b_+ indicate that the boundary is oriented. This reduces the generalized Stokes theorem to

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a),$$

which is actually the fundamental theorem of calculus!

Now, suppose that C is an oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$ and that D is the planar region bounded by C. If $\omega = A\,dx + B\,dy$ is a one-form on D, then

$$\begin{aligned}
\oint_C A\,dx + B\,dy &= \iint_D d(A\,dx + B\,dy) \\
&= \iint_D \left(\frac{\partial A}{\partial x}\,dx \wedge dx + \frac{\partial A}{\partial y}\,dy \wedge dx + \frac{\partial B}{\partial x}\,dx \wedge dy + \frac{\partial B}{\partial y}\,dy \wedge dy \right) \\
&= \iint_D \left(0 - \frac{\partial A}{\partial y}\,dx \wedge dy + \frac{\partial B}{\partial x}\,dx \wedge dy + 0 \right) \\
&= \iint_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy \\
&\equiv \iint_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx\,dy,
\end{aligned}$$

which is Green's theorem, relating the line integral around a closed curve to the double integral of the area it encloses.

We can extend this to the classical Stokes theorem, which relates the surface integral over an arbitrary surface in $\mathbb{R}^3$ to a line integral around the surface's boundary curve. To do this, let S be an oriented, piecewise-smooth surface in $\mathbb{R}^3$ bounded by an oriented, piecewise-smooth, simple closed curve $\mathcal{C}$ and let $\omega = A\,dx + B\,dy + C\,dz \in \Omega^1(S)$. Then,

$$\begin{aligned}
\oint_{\mathcal{C}} A\,dx + B\,dy + C\,dz &= \iint_S d(A\,dx + B\,dy + C\,dz) \\
&= \iint_S \frac{\partial A}{\partial x}\,dx \wedge dx + \frac{\partial A}{\partial y}\,dy \wedge dx + \frac{\partial A}{\partial Z}\,dz \wedge dx \\
&+ \frac{\partial B}{\partial y}\,dx \wedge dy + \frac{\partial B}{\partial y}\,dy \wedge dy + \frac{\partial B}{\partial y}\,dz \wedge dy \\
&+ \frac{\partial C}{\partial y}\,dx \wedge dz + \frac{\partial C}{\partial y}\,dy \wedge dz + \frac{\partial C}{\partial y}\,dz \wedge dz \\
&= \iint_S \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} \right) dz \wedge dx \\
&+ \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy.
\end{aligned}$$

If we define $\mathbf{F} := A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, then the integrand on the left-hand side is $\mathbf{F} \cdot d\mathbf{r}$, and the integrand on the right-hand side is $(\text{curl } \mathbf{F}) \cdot \mathbf{N} = (\nabla \times \mathbf{F}) \cdot \mathbf{N}$, where $\mathbf{N}$ is the unit normal vector pointing outward from S, and we arrive at

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\,dA,$$

which is Stokes' theorem as seen in vector calculus.

One can also show that the generalized Stokes theorem implies the fundamental theorem for line integrals and the divergence theorem. Therefore, we've come to the remarkable conclusion that all of these theorems seen in an introductory calculus course were amalgamated into a single elegant theorem in differential geometry.

7.3 Maxwell's Equations

Although the primary focus of this paper is how differential geometry relates to other mathematical disciplines, I'd like to briefly discuss an area where differential geometry arises in physics. In electrodynamics, Maxwell's field equations are (i) $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, (ii) $\nabla \cdot \mathbf{B} = 0$, (iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, and (iv) $\nabla \times \mathbf{E} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$. Here, $\mathbf{E}$ and $\mathbf{B}$ are the electric field and magnetic field, respectively; t is time, ρ is the charge per unit volume (charge density); $\mathbf{J}$ is the current per unit area (current density); and ϵ_0 and μ_0 represent fundamental constants, the permittivity and permeability of free space, respectively. These equations can be greatly simplified via differential geometry. First, we'll use Gaussian units instead of SI units to eliminate constants, and we'll also introduce a few more concepts. One is "Minkowski space," which is a four-dimensional space combining the three spatial components (x, y, x) with the time component t. The Minkowski space four-vector is defined as $-c\, dt \wedge dx \wedge dy \wedge dz$, and this space possesses a metric tensor of the form

$$g^{ij} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We also introduce the so-called "field two-form" $\mathbf{F} := \frac{1}{2} F_{ab}\, dx^a \wedge dx^b$ which describes both the electric and magnetic fields jointly via the *Faraday tensor*

$$F_{ab} := \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}.$$

When written in spacetime coordinates via the Minkowski metric, it becomes

$$\mathbf{F} = B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy + E_x\, dx \wedge dt + E_y\, dy \wedge dt + E_z\, dz \wedge dt,$$

and its Hodge dual is therefore

$$*\mathbf{F} = -B_x\, dx \wedge dt - B_y\, dy \wedge dt - B_z\, dz \wedge dt + E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy.$$

Lastly, we need the "current three-form" $\mathfrak{J} := \frac{1}{6} j^a \varepsilon_{abcd}\, dx^b \wedge dx^c \wedge dx^d$, which is given by

$$\mathfrak{J} = \rho\, dx \wedge dy \wedge dz - j^x\, dt \wedge dy \wedge dz - j^y\, dt \wedge dz \wedge dx - j^z\, dt \wedge dx \wedge dy,$$

where j^x, j^y, and j^z are the components of the current density.

With these quantities in hand, Maxwell's equations can be combined into

$$d\mathbf{F} = 0 \qquad \text{and} \qquad d(*\mathbf{F}) = \mathfrak{J}.$$

Referencing the original equations, (ii) and (iii) combined into the first equation, and (i) and (iv) combined into the second. Using the language of differential geometry simplified four equations involving four variables to just two equations in two variables. Thus, the many steps taken and quantities introduced to express Maxwell's equations in such a simple way allowed for a much more elegant formulation.

8 Conclusion

As I have learned from my research of various topics in differential geometry, this field provides alternative and generalized formulations of concepts from many mathematical disciplines. In fact, many of the terms seen in calculus and linear algebra, such as differentials, cross products, inner products, and the integral theorems of vector calculus can all be simplified by expressing them using the machinery of differential geometry. We have seen how tools like the wedge product, exterior derivative, Hodge star, metric tensor, volume form, and the musical isomorphisms can be combined to yield different results. Familiar geometric concepts such as length, area, and volume are all attainable via the metric tensor, a powerful tool on Riemannian manifolds which generalizes the inner product. The volume form gives us a direct approach to integrating functions on a manifold. Exterior derivatives are merely generalizations of the differential from calculus. These are just a few examples of many which demonstrate the power of differential geometry as it pertains to other mathematical fields. I have come to appreciate its unifying nature; ideas I once considered disparate and unrelated were in fact part of a larger framework I had yet to discover. The intuitive, integrated approach it offers to mathematics gives us a sophisticated method for linking different disciplines.

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