Illustration of how *Mathematica* works

Solving ODEs

In[1]:= `DSolve[y'[x] + 2 / x * y[x] == 1 / x^4 * y[x]^4, y[x], x]`

Out[1]= $\left\{\left\{y[x] \to -\frac{(-3)^{1/3}\, x}{\left(1 + 3\, x^9\, C[1]\right)^{1/3}}\right\}, \left\{y[x] \to \frac{3^{1/3}\, x}{\left(1 + 3\, x^9\, C[1]\right)^{1/3}}\right\}, \left\{y[x] \to \frac{(-1)^{2/3}\, 3^{1/3}\, x}{\left(1 + 3\, x^9\, C[1]\right)^{1/3}}\right\}\right\}$

Sometimes *Mathematica* writes things in an ugly way -
note that it wrote $(-3)^{1/3}$ and $(-1)^{2/3}$ which is not real numbers.
Here C[1] is the arbitrary constant; if the equation were of order 2,
then we would have two arbitrary constants, C[1] and C[2].

In[2]:= `DSolve[y'[x] == 2 * (y[x] + 5) * Exp[x], y[x], x]`

Out[2]= $\left\{\left\{y[x] \to -5 + e^{2\, e^x}\, C[1]\right\}\right\}$

In[3]:= `DSolve[y'[x] == (Sqrt[y[x]] - y[x]) / Sin[x], y[x], x]`

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Out[3]= $\left\{\left\{y[x] \to \frac{2\, e^{\frac{C[1]}{2}} \sqrt{\mathrm{Cos}\left[\frac{x}{2}\right]} + \frac{e^{C[1]}\, \mathrm{Cos}\left[\frac{x}{2}\right]}{\sqrt{\mathrm{Sin}\left[\frac{x}{2}\right]}} + \sqrt{\mathrm{Sin}\left[\frac{x}{2}\right]}}{\sqrt{\mathrm{Sin}\left[\frac{x}{2}\right]}}\right\}\right\}$

This is certainly not a pretty way to write the result.

Checking that a function is a solution of the ODE

In[51]:= `y1[x_] = (3 * x^3 / (1 + C * x^9))^(1 / 3)`

Out[51]= $3^{1/3} \left(\frac{x^3}{1 + C\, x^9}\right)^{1/3}$

In[52]:= `Simplify[y1'[x] + 2 / x * y1[x] - 1 / x^4 * y1[x]^4]`

Out[52]= 0

In[69]:= `y2[x_] = C * Exp[2 * Exp[x]] - 5`

Out[69]= $-5 + C\, e^{2\, e^x}$

In[70]:= `Simplify[y2'[x] - 2 * (y2[x] + 5) * Exp[x]]`

Out[70]= 0

In[66]:= `y3[x_] = (1 + C / Sqrt[Tan[x / 2]]) ^2`

Out[66]= $$\left(1+\frac{C}{\sqrt{\mathrm{Tan}\left[\frac{x}{2}\right]}}\right)^2$$

In[67]:= `Simplify[y3'[x] - (Sqrt[y3[x]] - y3[x]) / Sin[x]]`

Out[67]= $$\frac{1}{2}\,\mathrm{Csc}\left[\frac{x}{2}\right]\left(-\mathrm{Sec}\left[\frac{x}{2}\right]\left(-1+\sqrt{\mathrm{Cot}\left[\frac{x}{2}\right]\left(C+\sqrt{\mathrm{Tan}\left[\frac{x}{2}\right]}\right)^2}\right)+C\,\mathrm{Csc}\left[\frac{x}{2}\right]\sqrt{\mathrm{Tan}\left[\frac{x}{2}\right]}\right)$$

Mathematica needs some help, so I will simplify these expressions one by one:

In[68]:= `Simplify[y3'[x]]`

Out[68]= $$-\frac{1}{2}\,C\,\mathrm{Csc}\left[\frac{x}{2}\right]^2\left(C+\sqrt{\mathrm{Tan}\left[\frac{x}{2}\right]}\right)$$

In[64]:= `Simplify[- (Sqrt[y3[x]] - y3[x]) / Sin[x]]`

Out[64]= $$\mathrm{Csc}[x]\left(\left(1+\frac{C}{\sqrt{\mathrm{Tan}\left[\frac{x}{2}\right]}}\right)^2-\sqrt{\left(1+\frac{C}{\sqrt{\mathrm{Tan}\left[\frac{x}{2}\right]}}\right)^2}\right)$$

I will help *Mathematica* by telling it that Sqrt[A^2]=A - which, of course, is not always true, so it is good that *Mathematica* is cautious!

In[65]:= $$\mathbf{Simplify}\left[\mathbf{Csc}[x]\left(\left(1+\frac{C}{\sqrt{\mathbf{Tan}\left[\frac{x}{2}\right]}}\right)^2-\left(1+\frac{C}{\sqrt{\mathbf{Tan}\left[\frac{x}{2}\right]}}\right)\right)\right]$$

Out[65]= $$\frac{1}{2}\,C\,\mathrm{Csc}\left[\frac{x}{2}\right]^2\left(C+\sqrt{\mathrm{Tan}\left[\frac{x}{2}\right]}\right)$$

Comparing the expressions above, we see that y3[x] is indeed a solution of the ODE

Solving integrals

In[4]:= `Integrate[1 / (v^4 - 3 * v), v]`

Out[4]= $$-\frac{\mathrm{Log}[v]}{3}+\frac{1}{9}\,\mathrm{Log}\left[3-v^3\right]$$

In[50]:=
```
Integrate[1 / (Sqrt[y] - y), y]
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Out[50]= $-2\,\mathrm{Log}\left[1-\sqrt{y}\right]$

In[9]:=
```
Integrate[1 / Sin[x], x]
```

Out[9]= $-\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]\right]+\mathrm{Log}\left[\mathrm{Sin}\left[\frac{x}{2}\right]\right]$

In[57]:=
```
Integrate[Sqrt[Tan[x / 2]] / 2 / Sin[x], x]
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Out[57]= $\sqrt{\mathrm{Tan}\left[\frac{x}{2}\right]}$