$$\boxed{\frac{dy}{dx} + \frac{2}{x} y = \frac{1}{x^4} y^4}$$

1) Homogeneous: $v(x) = \frac{y(x)}{x}$, $y(x) = x\,v(x)$

$$x v'(x) + v(x) + 2v = v^4$$

$$x \frac{dv}{dx} = v^4 - 3v = v(v^3 - 3)$$

$$\int \frac{dv}{v(v^3-3)} = \int \frac{dx}{x};$$ use that $\frac{1}{v(v^3-3)} = -\frac{1}{3v} - \frac{v^2}{3(3-v^3)}$

$$-\frac{1}{3}\log v + \frac{1}{9}\log(3-v^3) = \log x + \log C$$

$$v^{-1/3} \cdot (3-v^3)^{1/9} = Cx$$

$$\left(v^{-3}(3-v^3)\right)^{\frac{1}{9}} = Cx$$

$$\left(\frac{3}{v^3} - 1\right)^{1/9} = Cx$$

$$\frac{3}{v^3} - 1 = Cx^9 \longrightarrow \frac{3}{v^3} = 1 + Cx^9$$

$$v = \left(\frac{3}{1+Cx^9}\right)^{1/3}$$

$$y = x\,v(x) = \left(\frac{3x^3}{1+Cx^9}\right)^{1/3}$$

2) <u>Bernoulli</u>: $v(x) = y(x)^{-3}$

$y(x) = v(x)^{-\frac{1}{3}}$, $y'(x) = -\frac{1}{3} v^{-\frac{4}{3}} v'$

$-\frac{1}{3} v^{-\frac{4}{3}} v' + \frac{2}{x} \frac{1}{v^{1/3}} = \frac{1}{x^4} \frac{1}{v^{4/3}} \quad \Big| \cdot v^{4/3}$

$-\frac{1}{3} v' + \frac{2}{x} v = \frac{1}{x^4}$

$v' - \frac{6}{x} v = -\frac{3}{x^4}$

$\rho(x) = e^{-\int \frac{6}{x} dx} = e^{-6 \ln x} = x^{-6}$

$\frac{d}{dx}\left(x^{-6} v(x)\right) = -\frac{3}{x^{10}}$

$x^{-6} v(x) = -3 \int x^{-10} dx = -3 \frac{x^{-9}}{-9} + C$

$= \frac{1}{3} x^{-9} + C$

$v(x) = \frac{1}{3} x^{-3} + C x^6$

$y(x) = \left(\frac{1}{3} x^{-3} + C x^6\right)^{-1/3} = \left(\frac{1 + 3C x^9}{3x^3}\right)^{-\frac{1}{3}}$

$$\boxed{\frac{dy}{dx} = 2(y+5)\,e^x}$$

1) <u>Separable:</u>

$$\int \frac{dy}{y+5} = \int 2e^x\,dx$$

$$\ln(y+5) = 2e^x + \ln C$$

$$y+5 = C\,e^{2e^x}$$

$$y(x) = -5 + C\,e^{2e^x}$$

2) <u>1^{st} order linear:</u>

$$\frac{dy}{dx} - 2e^x y = 10e^x$$

$$\rho(x) = e^{-2\int e^x dx} = e^{-2e^x}$$

$$\frac{d}{dx}\left[e^{-2e^x} y(x)\right] = 10\int e^{-2e^x} e^x\,dx$$

$$= 10\int e^{-2e^x} de^x = 10\int e^{-2\xi} d\xi$$

$$= 10\cdot\left(-\tfrac{1}{2}\right) e^{-2\xi} + C = -5\,e^{-2e^x} + C$$

$$y(x) = -5 + C\,e^{2e^x}$$

$$\boxed{\frac{dy}{dx} = \frac{\sqrt{y} - y}{\sin x}}$$

1) Separable: $\int \frac{dy}{\sqrt{y} - y} = \int \frac{dx}{\sin x}$

$$-2 \ln\left(1 - \sqrt{y}\right) = \ln \tan \frac{x}{2} + \ln C$$

$$\left(1 - \sqrt{y}\right)^{-2} = C \tan \frac{x}{2}$$

$$1 - \sqrt{y} = C \frac{1}{\sqrt{\tan \frac{x}{2}}}$$

$$y(x) = \left[1 + \frac{C}{\sqrt{\tan \frac{x}{2}}}\right]^2$$

$\boxed{\text{see below}}$

2) Bernoulli:

$$\frac{dy}{dx} + \frac{1}{\sin x} y = \frac{1}{\sin x} y^{1/2}$$

$$v = y^{1 - \frac{1}{2}} = \sqrt{y} \Rightarrow y = v^2$$

$$2v \frac{dv}{dx} + \frac{1}{\sin x} v^2 = \frac{1}{\sin x} v$$

$$\frac{dv}{dx} + \frac{1}{2 \sin x} v = \frac{1}{2 \sin x}$$

$$\rho(x) = e^{\int \frac{dx}{2 \sin x}} = e^{\frac{1}{2} \ln \tan \frac{x}{2}} = \sqrt{\tan \frac{x}{2}}$$

$$\frac{d}{dx}\left(\sqrt{\tan\frac{x}{2}}\; v(x)\right) = \frac{\sqrt{\tan\frac{x}{2}}}{2\sin x}$$

$$\sqrt{\tan\frac{x}{2}}\, v(x) = \int \frac{\sqrt{\tan\frac{x}{2}}}{2\sin x}\,dx = \sqrt{\tan\frac{x}{2}} + C$$

$$\Longrightarrow \quad v(x) = 1 + \frac{C}{\sqrt{\tan\frac{x}{2}}},$$

$$y(x) = \left(1 + \frac{C}{\sqrt{\tan\frac{x}{2}}}\right)^2$$

How to do the integrals:

- $$\int \frac{dy}{\sqrt{y}-y} = \int \frac{dy}{\sqrt{y}\,(1-\sqrt{y})} = 2\int \frac{d\sqrt{y}}{1-\sqrt{y}}$$

$$= -2\int \frac{d(1-\sqrt{y})}{1-\sqrt{y}} = -2\ln\left(1-\sqrt{y}\right)$$

- $$\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} \cdot \frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}$$

$$= \int \frac{1}{\left(\frac{\sin u}{\cos u}\right)} \frac{du}{\cos^2 u} \qquad \left(u = \frac{x}{2}\right)$$

$$= \int \frac{d(\tan u)}{\tan u} = \ln \tan\frac{x}{2} + C$$

- $$\int \frac{\sqrt{\tan\frac{x}{2}}}{2}\,\frac{dx}{\sin x} \underset{u=\frac{x}{2}}{=} \int \frac{d\tan u}{2\sqrt{\tan u}} = \sqrt{\tan\frac{x}{2}} + C$$