Derivation of the Fokker-Planck equation

Fokker-Planck equation is a partial differential equation for the transition density $\rho(x,t|y,s)$ of the stochastic process X_t satisfying the SDE

$$\mathrm{d}X_t = f(t,X_t)\,\mathrm{d}t + g(t,X_t)\,\mathrm{d}B_t\ , \tag{1}$$

where B_t is a Wiener process (and its generalized derivative, $\xi(t) = \mathrm{d}B_t/\mathrm{d}t$ is a Gaussian white noise). We discretize the SDE (1) as follows:

$$\Delta X_t = f(t,X_t)\,\Delta t + g(t,X_t)\,\Delta B_t\ , \tag{2}$$

where $\Delta X_t := X_{t+\Delta t} - X_t$ and $\Delta B_t := B_{t+\Delta t} - B_t$.
Preparation: using that $\mathbb{E}[\Delta B_t] = 0$ and $\mathbb{E}[(\Delta B_t)^2] = \Delta t$ and using the independence of the increments of the Wiener process, we obtain

$$\mathbb{E}\left[f(t,X_t)\Delta t | X_t = z\right] = \mathbb{E}\left[f(t,X_t)|X_t = z\right]\,\Delta t = f(t,z)\,\Delta t\ ; \tag{3}$$

$$\mathbb{E}\left[g(t,X_t)\Delta B_t | X_t = z\right] = g(t,z)\,\mathbb{E}\left[\Delta B_t|X_t = z\right] = g(t,z)\,\mathbb{E}\left[\Delta B_t\right] = 0\ ; \tag{4}$$

$$\begin{aligned}\mathbb{E}\left[g(t,X_t)^2(\Delta B_t)^2 | X_t = z\right] &= g(t,z)^2\,\mathbb{E}\left[(\Delta B_t)^2|X_t = z\right] \\ &= g(t,z)^2\,\mathbb{E}\left[(\Delta B_t)^2\right] = g(t,z)^2\,\Delta t\ ;\end{aligned} \tag{5}$$

using (3), (4) and (5), we can find the conditional moments of the jumps of X_t:

$$\mathbb{E}\left[\Delta X_t | X_t = z\right] = \mathbb{E}\left[f(t,X_t)\Delta t + g(t,X_t)\Delta B_t | X_t = z\right] = f(t,z)\,\Delta t\ ; \tag{6}$$

and

$$\begin{aligned}&\mathbb{E}\left[(\Delta X_t)^2 | X_t = z\right] \\ &\qquad = \mathbb{E}\left[f(t,X_t)^2\,(\Delta t)^2 + 2\,f(t,X_t)\,g(t,X_t)\,\Delta t\,\Delta B_t + g(t,X_t)^2\,(\Delta B_t)^2 | X_t = z\right] \\ &\qquad = g(t,z)^2\,\Delta t + o(\Delta t)\ ;\end{aligned} \tag{7}$$

note that these formulas can be rewritten as

$$\int (x-z)\,\rho(x,t+\Delta t|z,t)\,\mathrm{d}x = \mathbb{E}\left[X_{t+\Delta t} - X_t | X_t = z\right] = \mathbb{E}\left[\Delta X_t|X_t = z\right] = f(t,z)\,\Delta t\ , \tag{8}$$

and similarly,

$$\int (x-z)^2\,\rho(x,t+\Delta t|z,t)\,\mathrm{d}x = \mathbb{E}\left[(\Delta X_t)^2|X_t = z\right] = g(t,z)^2\,\Delta t + o(\Delta t)\ . \tag{9}$$

To derive the Fokker-Planck equation, we write the Chapman-Kolmogorov equation for $s < t$, and $\Delta t > 0$:

$$\rho(x,t+\Delta t|y,s) = \int \rho(x,t+\Delta t|z,t)\,\rho(z,t|y,s)\,\mathrm{d}z\ . \tag{10}$$

Multiply (10) by a smooth test function $R(x)$ and integrate both sides with respect to x to obtain the "smeared" Chapman-Kolmogorov equation

$$\int \mathrm{d}x\, R(x)\, \rho(x, t+\Delta t|y, s) = \int \mathrm{d}x\, R(x) \int \rho(x, t+\Delta t|z, t)\, \rho(z, t|y, s)\, \mathrm{d}z \ . \tag{11}$$

In the right-hand side of (11), expand $R(x)$ around z:

$$R(x) = R(z) + R'(z)\,(x-z) + \frac{1}{2} R''(z)\,(x-z)^2 + \cdots$$

then in the right-hand side of (10) we will have

$$\begin{aligned}
&\int R(x)\, \rho(x, t+\Delta t|z, t)\, \mathrm{d}x \\
&\quad = \int \left\{ R(z) + R'(z)\,(x-z) + \frac{1}{2} R''(z)\,(x-z)^2 + \cdots \right\} \rho(x, t+\Delta t|z, t)\, \mathrm{d}x \\
&\quad = R(z) \int \rho(x, t+\Delta t|z, t)\, \mathrm{d}x \\
&\qquad + R'(z) \int (x-z)\, \rho(x, t+\Delta t|z, t)\, \mathrm{d}x \\
&\qquad + R''(z) \int (x-z)^2\, \rho(x, t+\Delta t|z, t)\, \mathrm{d}x \\
&\quad = R(z) + R'(z)\, f(t,z)\, \Delta t + \frac{1}{2} R''(z)\, g(t,z)^2\, \Delta t + o(\Delta t) \ ,
\end{aligned} \tag{12}$$

where we have used the normalization $\int \rho(x, t+\Delta t|z, t)\, \mathrm{d}x = 1$ and the expressions (8) and (9). In the left-hand side of (11), we expand the short-time transition density, and then relabel the integration variable:

$$\begin{aligned}
\int R(x)\, \rho(x, t+\Delta t|y, s)\, \mathrm{d}x &= \int R(x)\, \left[\rho(x, t|y, s) + \partial_t \rho(x, t|y, s)\, \Delta t + o(\Delta t)\right] \mathrm{d}x \\
&= \int R(z)\, \rho(z, t|y, s)\, \mathrm{d}z + \Delta t \int R(z)\, \partial_t \rho(z, t|y, s)\, \mathrm{d}z + o(\Delta t) \ .
\end{aligned} \tag{13}$$

Now we plug (12) and (13) in the "smeared" Chapman-Kolmogorov equation (11) to obtain

$$\begin{aligned}
&\int R(z)\, \rho(z, t|y, s)\, \mathrm{d}z + \Delta t \int R(z)\, \partial_t \rho(z, t|y, s) + o(\Delta t)\, \mathrm{d}z \\
&\quad = \int R(z)\, \rho(z, t|y, s)\, \mathrm{d}z \\
&\qquad + \Delta t \int \left\{ R'(z)\, f(t,z) + \frac{1}{2} R''(z)\, g(t,z)^2 \right\} \rho(z, t|y, s)\, \mathrm{d}z + o(\Delta t) \ .
\end{aligned} \tag{14}$$

Canceling the equal terms in the left- and the right-hand side, collecting all the terms of order Δt and neglecting the terms of order $o(\Delta t)$, we obtain

$$0 = \int \left\{ R(z)\, \partial_t \rho(z, t|y, s) - \left[R'(z)\, f(t,z) + \frac{1}{2} R''(z)\, g(t,z)^2 \right] \rho(z, t|y, s) \right\} \mathrm{d}z \ .$$

Finally, we integrate the terms containing derivatives of $R(z)$ by parts to obtain

$$0 = \int R(z) \left\{ \partial_t \rho(z,t|y,s) + \partial_z \left[f(t,z)\, \rho(z,t|y,s) \right] - \frac{1}{2} \partial_{zz} \left[g(t,z)^2\, \rho(z,t|y,s) \right] \right\} \mathrm{d}z \ .$$

Since this equation holds for any choice of test function $R(z)$, we obtain the following equation for the transition density, which is called the Fokker-Planck equation:

$$\partial_t \rho(z,t|y,s) = -\partial_z \left[f(t,z)\, \rho(z,t|y,s) \right] + \frac{1}{2} \partial_{zz} \left[g(t,z)^2\, \rho(z,t|y,s) \right] \ , \tag{15}$$

which is often written in the form

$$\partial_t \rho(z,t|y,s) = \left[-\partial_z f(t,z) + \frac{1}{2} \partial_{zz} g(t,z)^2 \right] \rho(z,t|y,s) \ , \tag{16}$$

where it is understood that the differentiation with respect to z acts on everything that is to the right of it. The initial condition for the conditional density is

$$\rho(z,s|y,s) = \delta(z-y) \ .$$