Fokker-Planck equation is a partial differential equation for the transition density $\rho(x,t|y,s)$ of the stochastic process $X_t$ satisfying the SDE

$$dX_t = f(t, X_t) \, dt + g(t, X_t) \, dB_t, \quad (1)$$

where $B_t$ is a Wiener process (and its generalized derivative, $\xi(t) = dB_t/dt$ is a Gaussian white noise). We discretize the SDE (1) as follows:

$$\Delta X_t = f(t, X_t) \, \Delta t + g(t, X_t) \, \Delta B_t, \quad (2)$$

where $\Delta X_t := X_{t+\Delta t} - X_t$ and $\Delta B_t := B_{t+\Delta t} - B_t$.

Preparation: using that $\mathbb{E}[\Delta B_t] = 0$ and $\mathbb{E}[(\Delta B_t)^2] = \Delta t$ and using the independence of the increments of the Wiener process, we obtain

$$\mathbb{E} [f(t, X_t) \Delta t | X_t = z] = \mathbb{E} [f(t, X_t) | X_t = z] \, \Delta t = f(t, z) \, \Delta t; \quad (3)$$

$$\mathbb{E} [g(t, X_t) \Delta B_t | X_t = z] = g(t, z) \mathbb{E} [\Delta B_t | X_t = z] = g(t, z) \mathbb{E} [\Delta B_t] = 0; \quad (4)$$

$$\mathbb{E} [g(t, X_t)^2 (\Delta B_t)^2 | X_t = z] = g(t, z)^2 \mathbb{E} [(\Delta B_t)^2 | X_t = z]$$

$$= g(t, z)^2 \mathbb{E} [(\Delta B_t)^2] = g(t, z)^2 \Delta t; \quad (5)$$

using (3), (4) and (5), we can find the conditional moments of the jumps of $X_t$:

$$\mathbb{E} [\Delta X_t | X_t = z] = \mathbb{E} [f(t, X_t) \Delta t + g(t, X_t) \Delta B_t | X_t = z] = f(t, z) \, \Delta t; \quad (6)$$

and

$$\mathbb{E} [(\Delta X_t)^2 | X_t = z]$$

$$= \mathbb{E} [f(t, X_t)^2 (\Delta t)^2 + 2 f(t, X_t) g(t, X_t) \Delta t \Delta B_t + g(t, X_t)^2 (\Delta B_t)^2 | X_t = z]$$

$$= g(t, z)^2 \Delta t + o(\Delta t); \quad (7)$$

note that these formulas can be rewritten as

$$\int (x-z) \rho(x, t+\Delta t|z,t) \, dx = \mathbb{E} [X_{t+\Delta t} - X_t | X_t = z] = \mathbb{E} [\Delta X_t | X_t = z] = f(t, z) \, \Delta t, \quad (8)$$

and similarly,

$$\int (x-z)^2 \rho(x, t+\Delta t|z,t) \, dx = \mathbb{E} [(\Delta X_t)^2 | X_t = z] = g(t, z)^2 \Delta t + o(\Delta t). \quad (9)$$

To derive the Fokker-Planck equation, we write the Chapman-Kolmogorov equation for $s < t$, and $\Delta t > 0$:

$$\rho(x, t+\Delta t|y,s) = \int \rho(x, t+\Delta t|z,t) \rho(z,t|y,s) \, dz.$$

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Multiply (10) by a smooth test function $R(x)$ and integrate both sides with respect to $x$ to obtain the “smeared” Chapman-Kolmogorov equation

$$\int dx \ R(x) \rho(x, t + \Delta t|y, s) = \int dx \ R(x) \int \rho(x, t + \Delta t|z, t) \rho(z, t|y, s) \ dz . \quad (11)$$

In the right-hand side of (11), expand $R(x)$ around $z$:

$$R(x) = R(z) + R'(z) (x - z) + \frac{1}{2} R''(z) (x - z)^2 + \cdots$$

then in the right-hand side of (10) we will have

$$\int R(x) \rho(x, t + \Delta t|z, t) \ dx = \int \left\{ R(z) + R'(z) (x - z) + \frac{1}{2} R''(z) (x - z)^2 + \cdots \right\} \rho(x, t + \Delta t|z, t) \ dx$$

$$= R(z) \int \rho(x, t + \Delta t|z, t) \ dx + R'(z) \int (x - z) \rho(x, t + \Delta t|z, t) \ dx$$

$$+ R''(z) \int (x - z)^2 \rho(x, t + \Delta t|z, t) \ dx$$

$$= R(z) + R'(z) f(t, z) \Delta t + \frac{1}{2} R''(z) g(t, z)^2 \Delta t + o(\Delta t) , \quad (12)$$

where we have used the normalization $\int \rho(x, t + \Delta t|z, t) \ dx = 1$ and the expressions (8) and (9).

In the left-hand side of (11), we expand the short-time transition density, and then relabel the integration variable:

$$\int R(x) \rho(x, t + \Delta t|y, s) \ dx = \int R(x) \left[ \rho(x, t|y, s) + \partial_t \rho(x, t|y, s) \Delta t + o(\Delta t) \right] \ dx$$

$$= \int R(z) \rho(z, t|y, s) \ dz + \Delta t \int R(z) \partial_t \rho(z, t|y, s) \ dz + o(\Delta t) . \quad (13)$$

Now we plug (12) and (13) in the “smeared” Chapman-Kolmogorov equation (11) to obtain

$$\int R(z) \rho(z, t|y, s) \ dz + \Delta t \int \left\{ R'(z) f(t, z) + \frac{1}{2} R''(z) g(t, z)^2 \right\} \rho(z, t|y, s) \ dz + o(\Delta t) . \quad (14)$$

Canceling the equal terms in the left- and the right-hand side, collecting all the terms of order $\Delta t$ and neglecting the terms of order $o(\Delta t)$, we obtain

$$0 = \int \left\{ R(z) \partial_t \rho(z, t|y, s) - \left[ R'(z) f(t, z) + \frac{1}{2} R''(z) g(t, z)^2 \right] \rho(z, t|y, s) \right\} \ dz .$$
Finally, we integrate the terms containing derivatives of $R(z)$ by parts to obtain

$$0 = \int R(z) \left\{ \partial_t \rho(z, t|y, s) + \partial_z \left[ f(t, z) \rho(z, t|y, s) \right] - \frac{1}{2} \partial_{zz} \left[ g(t, z)^2 \rho(z, t|y, s) \right] \right\} \, dz .$$

Since this equation holds for any choice of test function $R(z)$, we obtain the following equation for the transition density, which is called the Fokker-Planck equation:

$$\partial_t \rho(z, t|y, s) = -\partial_z \left[ f(t, z) \rho(z, t|y, s) \right] + \frac{1}{2} \partial_{zz} \left[ g(t, z)^2 \rho(z, t|y, s) \right] , \quad (15)$$

which is often written in the form

$$\partial_t \rho(z, t|y, s) = \left[ -\partial_z f(t, z) + \frac{1}{2} \partial_{zz} g(t, z)^2 \right] \rho(z, t|y, s) , \quad (16)$$

where it is understood that the differentiation with respect to $z$ acts on everything that is to the right of it. The initial condition for the conditional density is

$$\rho(z, s|y, s) = \delta(z - y) .$$