MATH 2924 **Proof of Taylor's Theorem** **Fall 2015**

Notation: $I_n = \displaystyle\int_{t=a}^{x} f^{(n)}(t)\,\frac{(x-t)^{n-1}}{(n-1)!}\,\mathrm{d}t$

Elementary observations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{(x-t)^n}{n!}\right] = -\frac{1}{n!}\,n(x-t)^{n-1}(-1) = \frac{(x-t)^{n-1}}{(n-1)!} \tag{1}$$

$$I_{n+1} = \int_{t=a}^{x} f^{(n+1)}(t)\,\frac{(x-t)^n}{n!}\,\mathrm{d}t \tag{2}$$

$$I_1 = \int_{t=a}^{x} f^{(1)}(t)\,\frac{(x-t)^0}{0!}\,\mathrm{d}t = \int_{t=a}^{x} f'(t)\,\mathrm{d}t = f(x) - f(a) \qquad \text{by } \textbf{FTC} \tag{3}$$

Recursive relation between the I_n's:

$$
\begin{aligned}
I_n &= \int_{t=a}^{x} f^{(n)}(t)\,\frac{(x-t)^{n-1}}{(n-1)!}\,\mathrm{d}t \\
&\overset{(1)}{=} \int_{t=a}^{x} f^{(n)}(t)\,\mathrm{d}\left[-\frac{(x-t)^n}{n!}\right] \\
&\overset{\text{(by parts)}}{=} f^{(n)}(t)\left[-\frac{(x-t)^n}{n!}\right]\Bigg|_{t=a}^{x} - \int_{t=a}^{x}\left[-\frac{(x-t)^n}{n!}\right]\,\mathrm{d}\left[f^{(n)}(t)\right] \\
&= f^{(n)}(x)\left[-\frac{(x-x)^n}{n!}\right] - f^{(n)}(a)\left[-\frac{(x-a)^n}{n!}\right] - \int_{t=a}^{x} f^{(n+1)}(t)\left[-\frac{(x-t)^n}{n!}\right]\,\mathrm{d}t \\
&= 0 + \frac{f^{(n)}(a)}{n!}\,(x-a)^n + \int_{t=a}^{x} f^{(n+1)}(t)\,\frac{(x-t)^n}{n!}\,\mathrm{d}t \\
&\overset{(2)}{=} \frac{f^{(n)}(a)}{n!}\,(x-a)^n + I_{n+1}\ ,
\end{aligned}
$$

therefore

$$\boxed{I_n = \frac{f^{(n)}(a)}{n!}\,(x-a)^n + I_{n+1}} \tag{4}$$

Proof of Taylor's Theorem:

$$
\begin{aligned}
f(x) - f(a) \overset{(3)}{=} I_1 &\overset{(4)}{=} \frac{f'(a)}{1!}(x-a)^1 + I_2 \\
&\overset{(4)}{=} \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + I_3 \\
&\overset{(4)}{=} \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + I_4 \\
&\overset{(4)}{=} \dots \\
&\overset{(4)}{=} \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + I_{n+1} \\
&\overset{(2)}{=} \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_{t=a}^{x} f^{(n+1)}(t)\, \frac{(x-t)^n}{n!}\, \mathrm{d}t \ ,
\end{aligned}
$$

therefore

$$
\boxed{f(x) \ = \ \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_{t=a}^{x} f^{(n+1)}(t)\, \frac{(x-t)^n}{n!}\, \mathrm{d}t \ =: T_n(x) + R_n(x)}
$$

Upper bound on the remainder $R_n(x)$:

$$
\begin{aligned}
|R_n(x)| &= \left| \int_{t=a}^{x} f^{(n+1)}(t)\, \frac{(x-t)^n}{n!}\, \mathrm{d}t \right| \\
&= \int_{t=a}^{x} \left| f^{(n+1)}(t) \right| \left| \frac{(x-t)^n}{n!} \right| \mathrm{d}t \\
&\leq \max_{\xi \text{ between } a \text{ and } x} \left| f^{(n+1)}(\xi) \right| \int_{t=a}^{x} \left| \frac{(x-t)^n}{n!} \right| \mathrm{d}t \\
&= \max_{\xi \text{ between } a \text{ and } x} \left| f^{(n+1)}(\xi) \right| \left| \frac{(x-t)^{n+1}}{n!(n+1)} \right|_{t=a}^{x} \\
&= \max_{\xi \text{ between } a \text{ and } x} \left| f^{(n+1)}(\xi) \right| \frac{|x-a|^{n+1}}{(n+1)!} \ ,
\end{aligned}
$$

therefore

$$
\boxed{|R_n(x)| \leq \frac{M}{(n+1)!}\ |x-a|^{n+1} \ , \quad \text{where} \quad M = \max_{\xi \text{ between } a \text{ and } x} \left| f^{(n+1)}(\xi) \right|}
$$