Hints to some exercises from Section 7.1

Exercise 20. This is a *very* tricky one. Recall that

$$1+\tan^2 x = 1+\frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

to rewrite $\tan^2 x$ as $\dfrac{1}{\cos^2 x}-1$. Then the integral becomes

$$\int x\tan^2 x\,dx = \int x\left(\frac{1}{\cos^2 x}-1\right)dx = \int x\,\frac{dx}{\cos^2 x} - \int x\,dx\ .$$

For the first of these integrals use integration by parts with $u=x$, $v=\tan x$, and then substitution:

$$\int x\,\frac{dx}{\cos^2 x} = \int u\,dv = uv - \int v\,du = x\tan x - \int \tan x\,dx = x\tan x - \int \frac{\sin x}{\cos x}\,dx$$
$$= x\tan x - \int \frac{\sin x}{\cos x}\,dx = x\tan x + \int \frac{d\,\cos x}{\cos x} = x\tan x + \ln|\cos x| + C\ .$$

Putting all this together, we obtain

$$\int x\tan^2 x\,dx = x\tan x + \ln|\cos x| - \frac{x^2}{2} + C\ .$$

Checking the correctness of this formula by differentiating is quite amusing!

Exercise 22. You have to integrate by parts twice. For the first integration by parts set $u=\arcsin^2 x$, $v=x$, then $dv=dx$ and

$$\int \arcsin^2 x\,dx = \int u\,dv = uv - \int v\,du = x\,\arcsin^2 x - \int x\,\frac{2\arcsin x}{\sqrt{1-x^2}}\,dx\ ,$$

where we have used that, by the Chain Rule and the formula for the derivative of $\arcsin x$,

$$du = u'(x)\,dx = \frac{d}{dx}\left(\arcsin^2 x\right)dx = 2\arcsin x\,\frac{d}{dx}(\arcsin x)\,dx = 2\arcsin x\cdot\frac{1}{\sqrt{1-x^2}}\,dx\ .$$

Then notice that

$$-\int x\,\frac{2\arcsin x}{\sqrt{1-x^2}}\,dx = \int \frac{\arcsin x}{\sqrt{1-x^2}}\,d(1-x^2) = 2\int \arcsin x\,d\sqrt{1-x^2}\ ,$$

and in the last integral use integration by parts.

Exercise 26. Set $u=\ln y$, $v=2\sqrt{y}$, and integrate by parts.

Exercise 53. Write $\tan^n x$ as

$$\tan^n x = \tan^{n-2} x\tan^2 x = \tan^{n-2} x\frac{\sin^2 x}{\cos^2 x} = \tan^{n-2} x\frac{1-\cos^2 x}{\cos^2 x} = \tan^{n-2} x\left(\frac{1}{\cos^2 x}-1\right)\ .$$