The Kepler manifold as a Marsden-Weinstein reduced space

Hunter Ash
Advisor: Nikola Petrov

May 14, 2015

1 Introduction

The purpose of this paper is to give a mathematical treatment of the central force problem in classical and quantum mechanics. Our goal is to keep this discussion entirely general and geometric. Furthermore, we will include as many computations and proofs as possible. The two most salient benefits of a geometric approach to physics are that this perspective often yields new insights into old problems, and that this methodology is easily generalizable. We begin with a brief review of some Lie groups and algebras, and then address the Kepler problem.

2 Review of several Lie groups and algebras, some isomorphisms

Here we define a few specific Lie groups and establish some useful isomorphisms, following [3].

2.1 o(3)

Let the matrices δ_i, $i \in \{1, 2, 3\}$ denote the standard basis for the Lie algebra o(3) associated with the orthogonal group O(3). δ_i is defined by $(\delta_i)_{jk} = -\epsilon_{ijk}$

and may be though of as an infinitesimal rotation about the ith coordinate axis in $\mathbb{R}^3$. Because of the bilinearity of the Lie bracket, in this case the matrix commutator, the Lie bracket of a Lie algebra is completely determined by how it acts on the basis elements. In this case, $[\delta_i, \delta_j] = \epsilon_{ijk}\delta_k$.

The vector product on $\mathbb{R}^3$ is a Lie bracket, and it acts on the standard basis vectors as $e_i \times e_j = \epsilon_{ijk}\, e_k$. Thus, the map

$$a_1\delta_1 + a_2\delta_2 + a_3\delta_3 \mapsto (a_1, a_2, a_3) \in \mathbb{R}^3$$

is a Lie algebra isomorphism.

2.2 e(3)

e(3) is the Lie algebra of the Euclidean group E(3). The Euclidean group consists of all rigid transformations of $\mathbb{R}^3$, which may all be described as a rotation followed by a translation. Its action on $\mathbb{R}^3$ may be described as follows:

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

where $R \in \mathrm{SO}(3)$ and $x, v \in \mathbb{R}^3$. Since the tangent space to $\mathbb{R}^3$ is $\mathbb{R}^3$, we may identify e(3) with

$$\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a \in \mathrm{o}(3), b \in \mathbb{R}^3 \right\}$$

Together with our identification of o(3) with $\mathbb{R}^3$, this gives us an identification of e(3) with $\mathrm{o}(3) \times \mathrm{o}(3)$ as a vector space, though not as a Lie algebra. If we write $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} =: (a, b)$, then we can describe the Lie bracket of e(3) in terms of that of o(3):

$$[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2]_{\mathrm{o}(3)}, [a_1, b_2]_{\mathrm{o}(3)} - [a_2, b_1]_{\mathrm{o}(3)})$$

2.3 o(4)

Generally, the Lie algebra $\mathrm{o}(n)$ consists of all real antisymmetric $n \times n$ matrices. Thus, o(4) consists of all matrices of the form

$$\left\{ \begin{pmatrix} a & b \\ -b^T & 0 \end{pmatrix} \middle| a \in \mathrm{o}(3),\ b \in \mathbb{R}^3 \right\}$$

Once again we can identify o(4) as a vector space with o(3) × o(3), but in this case they are also isomorphic as Lie algebras. To see this, first note that the o(4) commutator may be written as follows in terms of the o(3) commutator:

$$[(a_1, b_1), (a_2, b_2)] = \Big([a_1, a_2]_{o(3)} + [b_1, b_2]_{o(3)},\ [a_1, b_2]_{o(3)} - [a_2, b_1]_{o(3)}\Big)$$

where we have identified the vectors b_i with elements of o(3) as above. Now consider the following subspaces of o(4):

$$\Delta = \{(a, a)\},\ \Delta' = \{(a, -a)\}$$

Both are closed under the Lie bracket. To see this for Δ, note that

$$[(a, a), (b, b)] = ([a, b]_{o(3)} + [a, b]_{o(3)},\ [a, b]_{o(3)} - [b, a]_{o(3)}) = 2([a, b]_{o(3)},\ [a, b]_{o(3)})$$

A similar computation shows the closure of Δ' under the bracket. Thus, Δ and Δ' are both not only subspaces but subalgebras of o(4), and together they clearly span the whole space. The Lie bracket of an element of Δ and an element of Δ' is 0:

$$[(a, a), (b, -b)] = ([a, b]_{o(3)} + [a, -b]_{o(3)},\ [a, -b]_{o(3)} - [b, a]_{o(3)}) = (0, 0)$$

Furthermore, both Δ and Δ' are isomorphic to o(3) under the map $\phi(a, \pm a) = 2a$:

$$\begin{aligned}\phi([(a, \pm a), (b, \pm b)]) &= \phi\big(2[a, b]_{o(3)},\ \pm 2[a, b]_{o(3)}\big)\\ &= 4[a, b]_{o(3)}\\ &= [2a, 2b]_{o(3)}\\ &= [\phi(a, \pm a),\ \phi(b, \pm b)]_{o(3)}\end{aligned}$$

Thus, $o(4) = \Delta \times \Delta' \cong o(3) \times o(3)$.

2.4 sl(2, ℝ) and sp(2, ℝ)

The group SL($n, \mathbb{R}$) consists of all real $n \times n$ matrices with determinant 1. The group Sp($2n, \mathbb{R}$) is the set of $2n \times 2n$ matrices M such that

$$M^T \Omega M = \Omega$$

where Ω is a fixed invertible antisymmetric matrix. Ω is typically chosen to be

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix, and we will follow this convention. Consider $\mathrm{Sp}(2, \mathbb{R})$. It consists of all 2×2 matrices $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Expanding the left hand side gives

$$\begin{pmatrix} 0 & ad - bc \\ bc - ad & 0 \end{pmatrix}$$

Thus, M is a symplectic matrix if and only if it has determinant 1. So in the special case $n = 2$, the special linear group $\mathrm{SL}(2, \mathbb{R})$ and the symplectic group $\mathrm{Sp}(2, \mathbb{R})$ are identified. This means that their Lie algebras $\mathrm{sl}(2, \mathbb{R})$ and $\mathrm{sp}(2, \mathbb{R})$ are also identified. For any matrix A, the following formula holds:

$$\det(\exp(A)) = \exp(\mathrm{tr}(A))$$

So in this case both Lie algebras consist of the traceless 2×2 real matrices.

3 The $\mathrm{SO}(4)$ symmetry of the Kepler problem

The Hamiltonian of the Kepler problem of classical mechanics is

$$H = \frac{\|p\|^2}{2m} + \frac{\alpha}{\|r\|}$$

where p is momentum, m is mass, r is position, and α is a constant. We will define two additional quantities and show that they are conserved: the *angular momentum*

$$L := r \times p$$

and the *Runge-Lenz vector*

$$F := p \times L + m\alpha \frac{r}{\|r\|}$$

The vector field generated by this Hamiltonian has integral curves satisfying

$$\dot{r} = \frac{p}{m}$$
$$\dot{p} = -\alpha \frac{r}{\|r\|^3}$$

Then along the flow generated by H, we have

$$\begin{aligned}\dot{L} &= \dot{r} \times p + r \times \dot{p} \\ &= \frac{1}{m} p \times p - \frac{\alpha}{\|r\|^3} r \times r \\ &= 0\end{aligned}$$

Similarly, repeated application of the Hamilton's equations can be used to show that F has zero time derivative, which is to say it is preserved along the flow generated by H. Equivalently, both L and F commute with H under the Poisson bracket.

We restrict ourselves to the case of bounded motion, i.e. $H < 0$. Define the following vectors in terms of their components:

$$\begin{aligned}M_i &= \frac{1}{2}\left(L_i + \frac{F_i}{\sqrt{-2mH}}\right) \\ N_i &= \frac{1}{2}\left(L_i - \frac{F_i}{\sqrt{-2mH}}\right)\end{aligned}$$

where $i \in \{1, 2, 3\}$. Since L, F, and H are conserved quantities, so are M and N. Repeated application of the relations $\{r_i, p_j\} = \delta_{i,j}$, $\{r_i, r_j\} = 0$, and $\{p_i, p_j\} = 0$ shows that the components of M and N obey the following commutation relations under the Poisson bracket:

$$\begin{aligned}\{M_i, M_j\} &= \epsilon_{ijk} M_k \\ \{N_i, N_j\} &= \epsilon_{ijk} N_k \\ \{M_i, N_j\} &= 0\end{aligned}$$

These are the same commutation relations obeyed by the standard basis elements of $\mathrm{o}(3) \times \mathrm{o}(3)$. Thus, the six-dimensional space spanned by the M_i and N_i is Lie algebra isomorphic to $\mathrm{o}(4)$. Since H commutes with all the M_i and N_i, this establishes the SO(4) symmetry of the Kepler problem.

4 The Kepler Manifold

Here we describe the geometry of the set of bound (negative total energy) orbits of the Kepler problem. Denoted T^+S^3, it is called the Kepler manifold.

We also discuss the problem of the completion of the Hamiltonian vector field, and the regularization of the Kepler manifold. The problem is this: the collision orbits, those with zero angular momentum and which pass through the origin, are not well-defined because the velocity diverges as the particle approaches the origin. The problem can be resolved by replacing time with another parameter. With this choice of parameter, the Hamiltonian flow of the Kepler problem is mapped symplectomorphically onto the unit geodesic flow on the 3-sphere S^3. The problematic collision orbits are mapped onto those geodesics passing through the "north pole" of S^3, which is defined in terms of the stereographic projection involved in the mapping. In this new description, we may complete the set of Kepler orbits. We will follow the approach of [2].

In this section we will use specialized notation for distinguishing the various vectors and forms according to their number of coordinates. Letters with an arrow above them ($\vec{v}$) denote vectors or forms with 4 components. Plain letters with no subscripts denote vectors or forms with 3 components (v). Letters with subscripts or superscripts denote real numbers (v^i).

4.1 Stereographic projection

We define stereographic projection as follows: S^3 is the subset of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{aligned} \vec{a} &= (a^0, a) \\ a &= (a^1, a^2, a^3) \in \mathbb{R}^3 \\ \|\vec{a}\| &= a^0 \cdot a^0 + \|a\| = 1 \end{aligned}$$

We denote by S_N the sphere S^3 with the north pole $(1,0,0,0)$ removed. Stereographic projection is the map

$$\begin{aligned} \text{Ster} : S_N &\to \mathbb{R}^3 \\ \vec{a} &\mapsto \frac{a}{1-a^0} \end{aligned}$$

We will also refer to the image under Ster of the vector $\vec{a}$ as b. The inverse of this map is given by

$$\text{Ster}^{-1}(b) = \vec{a} = \left(\frac{\|b\|^2 - 1}{\|b\|^2 + 1}, \frac{2b}{\|b\|^2 + 1} \right)$$

4.2 The canonical 1-form

The canonical 1-form on $T^*\mathbb{R}^3$ is defined as follows. Let $b \in \mathbb{R}^3$ and $B_b \in T^*_b\mathbb{R}^3$. Then (b, B_b) is a point in $T^*\mathbb{R}^3$. Let

$$B_b = B_i(b)db^i$$

where $i \in \{1, 2, 3\}$ and let $X_{(b,B_b)} \in T_{(b,B_b)}(T^*\mathbb{R}^3)$ given by

$$\begin{aligned} X_{(b,B_b)} &= X^i(b, B_b)\frac{\partial}{\partial b^i}\Big|_{(b,B_b)} \\ &\quad + \tilde{X}_i(b, B_b)\frac{\partial}{\partial B_i}\Big|_{(b,B_b)} \end{aligned}$$

Then the canonical one form $\Theta^{\mathbb{R}^3}$ at the point (b, B_b) takes the vector $X_{(b,B_b)}$ to

$$\Theta^{\mathbb{R}^3}_{(b,B_b)}(X_{(b,B_b)}) := B_i(b)X^i(b, B)$$

In other words,

$$\Theta^{\mathbb{R}^3}_{(b,B_b)} = B_i(b)db^i$$

We can carry out a similar construction for T^*S_N by first doing so for $T^*\mathbb{R}^4$ and restricting it to the sphere by insisting that $\|\vec{a}\| = 1$ and, if $\vec{A}_{\vec{a}} = A_\mu(\vec{a})\, da^\mu,\ \mu \in \{0, 1, 2, 3\}$, then

$$\sum_\mu a^\mu A_\mu = 0$$

The pullback Ster^* of our map Ster is a diffeomorphism $T^*\mathbb{R}^3 \to T^*S_N$ which maps the canonical 1-form $\Theta^{\mathbb{R}^3}$ on $T^*\mathbb{R}^3$ to the canonical 1-form Θ^{S_N} on the punctured 3-sphere S_N. Recall that $b := \mathrm{Ster}(\vec{a})$, that $\vec{a} = (a^0, a)$ and that $\vec{A} = A_0\, da^0 + \sum_i A_i\, da^i$. In terms of components, Ster relates $T^*\mathbb{R}^3$ to T^*S_N as follows:

$$\begin{aligned} B_i(b) &= (1 - a^0)A_i(\vec{a}) + A_0(\vec{a})a^i \\ db^i &= \frac{da^i}{1 - a^0} + \frac{a^i\, da^0}{(1 - a^0)^2} \\ A_0(\vec{a}) &= \sum_i B_i(b)b^i \\ A_i(\vec{a}) &= \frac{1}{2}(1 + \|b\|^2) - \left(\sum_i B_i(b)b^i\right) b^i \end{aligned}$$

From these we obtain the relation

$$\|\vec{A}_{\vec{a}}\| = (A_0(\vec{a}))^2 + \|A_{\vec{a}}\|^2 = \frac{1}{4}(1+\|b\|^2)^2\|B_b\|^2$$

Thus, the geodesic kinetic energy function on T^*S^3

$$G(\vec{A}_{\vec{a}}) = \frac{1}{2}\|\vec{A}_{\vec{a}}\|^2$$

is mapped to the following function on $T^*\mathbb{R}^3$

$$K(b, B_b) := G(\vec{A}_{\vec{a}}) = \frac{1}{8}(1+\|b\|^2)\|B_b\|^2$$

4.3 The Hamiltonian vector field as geodesic flow on S^3

Define

$$\begin{aligned} u : \mathbb{R}_{>0} &\to \mathbb{R} \\ x &\mapsto \sqrt{2x} - 1 \end{aligned}$$

and let ξ_K be the Hamiltonian vector field generated by K, i.e. such that $\omega(\xi_K, \cdot) = dK$. Then by the chain rule,

$$\xi_{u(K)} = u'(K)\xi_K = \frac{1}{\sqrt{2K}}\xi_K$$

Note that on the surface $K^{-1}(\frac{1}{2})$, we have $\xi_K = \xi_{u(K)}$. On all level sets of K satisfying $H < 0$, both vector fields give the geodesic flow (though not necessarily the unit geodesic flow) of the sphere, described in stereographically projected coordinates. Generally, we have

$$\begin{aligned} u(K(b, B_b)) &= \sqrt{2(\frac{1}{8}(1+\|b\|^2)^2\|B_b\|^2)} - 1 \\ &= \frac{1}{2}(1+\|b\|^2)\|B_b\| - 1 \end{aligned}$$

So $u(K)$ is the Kepler Hamiltonian multiplied by $\|B\|$ (with an added constant), if we set $b = p$ and $B_b = -r$. Crucially, we must assume $\|B_b\| \neq 0$.

$$\begin{aligned}\frac{1}{\|B_b\|} u(K) &= \frac{1}{2}(1 + \|b\|^2) - \frac{1}{\|B_b\|} \\ &= \frac{\|b\|^2}{2} - \frac{1}{\|B\|} + \frac{1}{2} \\ &= H + \frac{1}{2}\end{aligned}$$

Defining $f(b, B) = \|B_b\|$, we have

$$u(K) = f(b, B_b)\left(H + \frac{1}{2}\right)$$

On the surface $u(K) = 0$, equivalently $K = \frac{1}{2}$ or $H = -\frac{1}{2}$, we have

$$\begin{aligned}du(K(b, B_b)) &= d\left(f(H + 1/2)\right) \\ &= (df) \cdot (H + 1/2) + f \cdot d(H + 1/2) \\ &= f \cdot dH\end{aligned}$$

Recall that $\xi_{u(K)}$ is defined to be the vector field such that $\omega(\xi_{u(K)}, \cdot) = d(u(K))$ and ξ_H the vector field such that $\omega(\xi_H, \cdot) = dH$. But $d(u(K)) = f \cdot dH$ and symplectic forms are bilinear, so

$$\xi_{u(K)} = f \cdot \xi_H$$

Recall that $K(b, B_b) := G(\vec{A}) = \frac{1}{2}\|\vec{A}\|^2$ so that $K = \frac{1}{2} \Rightarrow \|\vec{A}\|^2 = 1$. Also recall that, on this surface, $\xi_{u(K)}$ gives the unit geodesic flow on the sphere. Thus, we have diffeomorphically mapped the Hamiltonian flow of the Kepler problem to the unit geodesic flow on the 3-sphere, for orbits of energy $H = -\frac{1}{2}$ and under the condition that $\|B\| \neq 0$. We could perform a similar calculation for any other negative value of H, mapping it to the geodesic flow on the 3-sphere but with a different speed (orbiting faster or slower). Thus, the space of orbits of the Kepler problem may be identified with the set of all nonzero covector fields on the 3-sphere. This manifold is denoted T^+S^3 and is called the Kepler manifold.

Because the geodesic flow on T^*S^3 is obviously complete, this identification allows us to complete the Hamiltonian vector field of the Kepler problem.

5 The Kepler manifold as a Marsden-Weinstein reduced space

Here we show how to realize the Kepler manifold T^+S^3 as a Marsden-Weinstein reduced space of the Lie algebra $\mathrm{sp}(2,\mathbb{R}) \cong \mathrm{sl}(2,\mathbb{R})$.

5.1 The Kepler manifold as a subset of $\Lambda^2\,\mathbb{R}^6$

First we give an alternative construction of T^+S^3. Let $\mathbb{R}^6$ be a 6-six dimensional real vector space equipped with a scalar product of signature $(+,+,-,-,-,-)$. That is, for any two vectors $a = (a^{-1},...,a^4)$ and $b = (b^{-1},...,b^4)$,

$$(a,b) = a^{-1}b^{-1} + a^0b^0 - \left(\sum_{i=1}^{4} a^ib^i\right)$$

Define

$$\mathcal{K} := \{u \wedge v \neq 0 \,\Big|\, \|u\| = \|v\| = (u,v) = 0\} \subset \Lambda^2\,\mathbb{R}^6$$

Since the orthogonal group $\mathrm{O}(2,4)$ is defined to be the group whose action does not change the value of this scalar product, and $u \wedge v = 0$ iff $u = 0$, $v = 0$, or $u = \lambda v$ for $\lambda \in \mathbb{R}$, it is clear the $\mathcal{K}$ is invariant under the action of $\mathrm{O}(2,4)$.

We will show that $\mathcal{K}$ is in fact an orbit of $\mathrm{O}(2,4)$, and that under the action of the connected component $\mathrm{O}_0(2,4)$ of $\mathrm{O}(2,4)$, $\mathcal{K}$ decomposes into the union of two orbits, $\mathcal{K} = \mathcal{K}_+ \cup \mathcal{K}_-$. Each of these two orbits is diffeomorphic to the Kepler manifold T^+S^3.

First we choose an orthogonal basis $(e^{-1}, e^0, e^1, e^2, e^3, e^4)$ of $\mathbb{R}^6$ such that

$$\|e^{-1}\| = \|e^0\| = 1$$
$$\|e^i\| = -1,\ i \in \{1,2,3,4\}$$

This gives us a splitting of $\mathbb{R}^6$ into orthogonal subspaces: $\mathbb{R}^2$ spanned by e^{-1}, e^0 which inherits a positive definite scalar product, and $\mathbb{R}^4$ spanned by e^1, e^2, e^3, e^4 which inherits a negative definite scalar product.

Next we define a scalar product on $\Lambda^2\,\mathbb{R}^6$ in terms of the scalar product on $\mathbb{R}^6$ described above:

$$(u \wedge v, x \wedge y) := \det \begin{pmatrix} (u,x) & (u,y) \\ (v,x) & (v,y) \end{pmatrix}$$

Now for all $u \wedge v \in \mathcal{K}$, we must have

$$(e^{-1} \wedge e^0) \cdot (u \wedge v) \neq 0$$

To see this note that

$$(e^{-1} \wedge e^0) \cdot (e^\mu \wedge e^\nu) = (e^{-1}, e^\mu)(e^0, e^\nu) - (e^{-1}, e^\nu)(e^0, e^\mu) \neq 0$$

If and only if $\mu = -1$ and $\nu = 0$ or vice versa. Since the dot product is bilinear, this means that $(e^{-1} \wedge e^0) \cdot (u \wedge v) \neq 0$ if and only if u has a nonzero e^{-1} component and v has a nonzero e^0 component or the other way round. We are assured that this is the case, as both u and v have 0 $\mathbb{R}^6$-norm, are not the zero vector, and are not parallel, and if they only had components in the negative definite subspace this would not be possible. Thus, we have the following decomposition of $\mathcal{K}$:

$$\mathcal{K} = \mathcal{K}_+ \cup \mathcal{K}_-$$
$$\mathcal{K}_+ = \{u \wedge v \in \mathcal{K} \big| (e^{-1} \wedge e^0) \cdot (u \wedge v) > 0\}$$
$$\mathcal{K}_- = \{u \wedge v \in \mathcal{K} \big| (e^{-1} \wedge e^0) \cdot (u \wedge v) < 0\}$$

We now parameterize $\mathcal{K}_+$. First, we choose a new u and v from the space spanned by the original u and v such that

$$(e^0, u) = (e^1, v) = 0$$

and in a way that does not change their wedge product $u \wedge v$. We can do this as follows: recall that $u \wedge v = u \wedge (v + ru)$ for $r \in \mathbb{R}$ and similarly $u \wedge v = (u + sv) \wedge v$, $s \in \mathbb{R}$. We choose our r and s so that the resulting two factors are as desired. Since this process did not alter the wedge product, we continue to refer to our new choices as u and v. We can further arrange for (e^{-1}, u) and (e^0, v) to be positive: recall we are in $\mathcal{K}_+$ and with our current choice of u and v, $(e^{-1} \wedge e^0) \cdot (u \wedge v) = u^{-1} v^0 > 0$ so if (e^{-1}, u) is negative then so must be (e^0, v). But we have

$$u \wedge v = (-1)(-1) u \wedge v = (-u) \wedge (-v)$$

so we can always choose our u and v as desired.

We have established that neither u nor v can lie entirely in either of our subspaces $\mathbb{R}^2$ and $\mathbb{R}^4$, so they must be of the form

$$u = u^{-1} e^{-1} + u_-$$
$$v = v^0 e^0 + v_-$$

where $u_-, v_- \in \mathbb{R}^4\backslash\{0\}$. Since $u \wedge v$ is only dependent on the product of the magnitudes of the factors rather than their individual magnitudes, only the ratios

$$\frac{u^{-1}}{\|u_-\|}$$
$$\frac{v^0}{\|v_-\|}$$

are fixed, so we can arrange for $\|u_-\| = \|v_-\| = -1$. Furthermore, we must have $(u_-, v_-) = 0$. To see this, recall that if $u = u_+ + u_-$, $u_+ \in \mathbb{R}^2$, $u_- \in \mathbb{R}^4$ and similarly for v, their $\mathbb{R}^6$ inner product (signature $(+, +, -, -, -, -)$) may be expressed in terms of the standard Euclidean inner products on $\mathbb{R}^2$ and $\mathbb{R}^4$ as follows:

$$(u, v)_{\mathbb{R}^6} = (u_+, v_+)_{\mathbb{R}^2} - (u_-, v_-)_{\mathbb{R}^4}$$

We have chosen u and v such that $(u, v)_{\mathbb{R}^6} = 0$ and $(u_+, v_+)_{\mathbb{R}^2} = 0$. Thus, we must have that u_- and v_- are orthogonal.

To summarize, we have parameterized $\mathcal{K}_+$ as follows:

$$u \wedge v = s(e^{-1} + u') \wedge (e^0 + v')$$
$$\|u_-\| = \|v_-\| = -1$$
$$(u_-, v_-) = 0$$

with $s \in \mathbb{R}^\times$. Note that u_-, being in $\mathbb{R}^4$ and having (negative) unit norm, naturally lives on the 3-sphere. v_- lives in $\mathbb{R}^4$, must be orthogonal to u (and so must live in a subspace isomorphic to $\mathbb{R}^3$) and has unit norm (restricting it to S^2). This gives us an explicit diffeomorphism between K_+ and the Kepler manifold:

$$\mathcal{K}_+ \to \mathbb{R}^\times \times T^1S^3 \cong T^+S^3$$

where T^1S^3 denotes the bundle of unit covectors. An analogous calculation can be performed for $\mathcal{K}_-$.

Now we shall use the realization of the Kepler manifold as $\mathcal{K}$ in order to express it as a Marsden-Weinstein reduced space.

5.2 Some isomorphisms

First, we will establish several useful isomorphisms. Let V be a real vector space with a non-degenerate scalar product. Then we can identify the vector space of 2-vectors $\Lambda^2(V)$ with the Lie algebra $\mathrm{o}(v)$ of the orthogonal group of V, consisting of the real antisymmetric matrices, as

$$(u \wedge v)x = (u, x)v - (v, x)u$$

The antisymmetry of such an operator is obvious, and its linearity follows from the bilinearity of the scalar product. The Killing form $\langle A, B\rangle := \mathrm{tr}(adA \circ adB)$ defines an nondegenerate inner product on $\mathrm{o}(V)$ and therefore gives us an identification of $\mathrm{o}(V)$ with its dual space $\mathrm{o}(V)^*$, so we have

$$\mathrm{o}(V)^* \cong \mathrm{o}(V) \cong \Lambda^2(V)$$

Under this identification, the coadjoint action of $\mathrm{O}(V)$ on $\mathrm{o}(v)^*$ is its usual action on $\Lambda^2(V)$:

$$g(u \wedge v) = gu \wedge gv$$

Let W be a real symplectic vector space with symplectic form ω and corresponding Poisson bracket $\{\cdot, \cdot\}$. Again using the Killing form as an inner product, we may identify $\mathrm{sp}(W)$ with its dual space $\mathrm{sp}(W)^*$. We can also identify $\mathrm{sp}(W)$ with the space of homogeneous quadratic polynomials via the map

$$P(A)(w) = \frac{1}{2}\omega(Aw, w) = \frac{1}{2}\omega_{ij}A^i_k w^k w^j$$

We will show that this map is a Lie algebra isomorphism between $\mathrm{sp}(W)$ under the commutator and the space of homogeneous quadratic polynomials under the Poisson bracket. It is obviously a vector space isomorphism so we need only check that it preserves the Lie bracket. First, some notation. Since W is a symplectic vector space, it is even dimensional (dimension $2n$), and so the indices i, j, k in the definition of P range from 1 to $2n$. For a given vector w, we will identify its first n components with the $\mathbb{R}^n$-vector x and the second n components with y, and assume that these coordinates are canonical. It is easy to check from the definition of $\mathrm{Sp}(W)$ that any matrix $A \in \mathrm{sp}(W)$ must be of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

where the lower case letters in this instance represent arbitrary $n \times n$ block matrices with real entries. In fact, matrices of this form exactly comprise sp(W). This allows us to write $P(A)$ as

$$P(A) = \frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Take a second matrix from sp(W)

$$B = \begin{pmatrix} q & r \\ s & -q \end{pmatrix}$$

and apply the general formula

$$\{P(A), P(B)\} = \sum_i \frac{\partial P(A)}{\partial x^i} \frac{\partial P(B)}{\partial y^i} - \frac{\partial P(A)}{\partial y^i} \frac{\partial P(B)}{\partial x^i}$$

to check directly that

$$\{P(A), P(B)\} = P([A, B])$$

This establishes that P is a Lie algebra isomorphism.

By letting A vary, the map P identifies sp$(W)^*$ with $S^2(W)$, the space of symmetric 2-tensors on W:

$$S^2(W) \to \mathrm{sp}(W)^*$$
$$w \otimes w \mapsto \frac{1}{2}\omega_{ij}(\cdot)^i_k w^k w^j$$

Under this identification, the coadjoint action of Sp(W) is its usual action $\phi : \mathrm{Sp}(W) \times S^2(W) \to S^2(W)$ on $S^2(W)$. To see this, recall that the coadjoint action of an element g on a 1-form μ is defined by

$$\langle Ad^\sharp_g \mu, B\rangle := \langle \mu, Ad_{g^{-1}} B\rangle$$

and note that

$$g(w \otimes w) := gw \otimes gw \mapsto \frac{1}{2}\omega((\cdot)gw, gw)$$

We can multiply both arguments of ω by g^{-1} on the left without changing the value of ω (this is the definition of the symplectic group):

$$\frac{1}{2}\omega((\cdot)gw, gw) = \frac{1}{2}\omega(g^{-1}(\cdot)gw, w)$$

Plugging in a matrix $B \in \mathrm{sp}(W)$ we see that we have the coadjoint action:

$$\begin{aligned}(Ad_g^\sharp P(w))(B) &= \frac{1}{2}\omega(Bgw, gw) \\ &= \frac{1}{2}\omega(g^{-1}Bgw, w) \\ &= P(w)(Ad_{g^{-1}}B)\end{aligned}$$

So we have the following commutative diagram:

$$\begin{array}{ccc} S^2(W) & \xrightarrow{P} & \mathrm{sp}(W)^* \\ {\scriptstyle \phi_g}\downarrow & & \downarrow{\scriptstyle Ad_g^\sharp} \\ S^2(W) & \xrightarrow{P} & \mathrm{sp}(W)^* \end{array}$$

With this identification, the moment map for the group action of $\mathrm{Sp}(W)$ becomes the square, as we will show.

$$\begin{aligned} sq : W &\to S^2(W) \cong \mathrm{sp}(W)^* \\ w &\mapsto w \otimes w \sim \frac{1}{2}\omega_{ij}(\cdot)^i_k w^k w^j \end{aligned}$$

Let V be a vector space with a scalar product $(\cdot,\cdot)_V$ and W be a symplectic vector space with symplectic form $(\cdot,\cdot)_W$. Then the product vector space $V \otimes W$ can be endowed with a natural symplectic form $(\cdot,\cdot)_{V\otimes W}$. We define it for elements of the form $v \otimes w$ and extend it to the rest of $V \otimes W$ by bilinearity:

$$(v_1 \otimes w_1, v_2 \otimes w_2)_{V\otimes W} := (v_1, v_2)_V (w_1, w_2)_W \tag{1}$$

This choice of symplectic form gives us a natural embedding of $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$ into $\mathrm{Sp}(V \otimes W)$:

$$\begin{aligned} \mathrm{O}(V) &\to \mathrm{Sp}(V \otimes W) \\ A &\mapsto A \otimes I_W \end{aligned}$$

$$\begin{aligned} \mathrm{Sp}(W) &\to \mathrm{Sp}(V \otimes W) \\ B &\mapsto I_V \otimes B \end{aligned}$$

It is easy to check that, under this embedding, $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$ are each others' centralizers:

$$(A \otimes I_W)(I_V \otimes B) = (AI_V \otimes I_W B) = (A \otimes B) = (I_V A \otimes BI_W) = (I_V \otimes B)(A \otimes I_W)$$

5.3 Constructing the moment map of $\mathrm{Sp}(W)$

Here we construct the moment maps for the action of $\mathrm{Sp}(W)$ on W and the action of $\mathrm{O}(V)$ on V directly from the definition, which can be found in the appendix.

5.3.1 Notation

Let (W, ω) be a symplectic vector space, and

$$\Psi : \mathrm{Sp}(W) \times W \to W \tag{2}$$

be the representation of the symplectic group on (W, ω). To simplify the notations, we will often write gw instead of $\Psi_g w$ (where $w \in W$), and will think of g as a matrix acting on the column vector $w \in W$:

$$gw := \Psi_g w \ .$$

The symplectic group $\mathrm{Sp}(W)$ is characterized by

$$\begin{aligned} \mathrm{Sp}(W) &= \{g \in \mathrm{GL}(W) \ : \ \omega(gw, gw') = \omega(w, w') \ \forall \, w, w' \in W\} \\ &= \{g \in \mathrm{GL}(W) \ : g^T \omega g = \omega\} \ . \end{aligned}$$

To obtain its Lie algebra $\mathrm{sp}(W)$, set $g = e^{tA}$, differentiate $\omega(e^{tA}w, e^{tA}w') = \omega(w, w')$ with respect to t and set $t = 0$ to obtain the condition

$$\omega(Aw, w') + \omega(w, Aw') = 0 \ , \tag{3}$$

or, equivalently,

$$\begin{aligned} \mathrm{sp}(W) &= \{A \in \mathrm{gl}(W) \ : \ e^{tA^T} \omega e^{tA} = \omega\} \\ &= \{A \in \mathrm{gl}(W) \ : A^T \omega + \omega A = 0\} \\ &= \{A \in \mathrm{gl}(W) \ : A^T = \omega A \omega\} \ . \end{aligned}$$

The action (2) induces a natural action of $\mathrm{Sp}(W)$ on $S^k(W)$:

$$\Psi^k : \mathrm{Sp}(W) \times S^k(W) \to S^k(W) : (g, \otimes^k w) \mapsto \Psi^k_g(\otimes^k w) := \otimes^k (gw) \ , \qquad (4)$$

where $\otimes^k w := w \otimes w \otimes \cdots \otimes w$ (k times). Using this, we can define a natural action of the Lie algebra $\mathrm{sp}(W)$ on $S^k(W)$:

$$\psi^k : \mathrm{sp}(W) \times S^k(W) \to S^k(W) : (A, \otimes^k w) \mapsto \psi^k_A(\otimes^k w) \ , \qquad (5)$$

where

$$\psi^k_A(\otimes^k w) := \left.\frac{d}{dt}\right|_{t=0} \Psi^k_{\exp(tA)}(\otimes^k w) = \left.\frac{d}{dt}\right|_{t=0} \otimes^k (e^{tA} w) \ . \qquad (6)$$

5.3.2 $\mathrm{Sp}(W)$-intertwining isomorphism $S^2(W)^* \to P^2(W)$

Let $P^2(W)$ be the vector space (over $\mathbb{R}$) of real-valued homogeneous quadratic polynomials, i.e., $p \in P^2(W)$ if

$$p(w) = \frac{1}{2}\, \underline{p}(w, w) \ , \qquad w \in W \ , \qquad (7)$$

where $\underline{p}$ is a symmetric bilinear form on W:

$$\underline{p}(w, w') := \underline{p}_{ij} w^i (w')^j = w^T \underline{p} w' \ , \qquad \underline{p}_{ij} = \underline{p}_{ji} \ .$$

Clearly, $\underline{p}$ can be considered as an element of $S^2(W)^*$:

$$\underline{p} : S^2(W) \to \mathbb{R} : w \otimes w \mapsto \langle \underline{p}, w \otimes w \rangle := \underline{p}(w, w) \ .$$

This establishes a bijective correspondence between $S^2(W)^*$ and $P^2(W)$ as vector spaces.

The action Ψ^2 of $\mathrm{Sp}(W)$ on $S^2(W)$ defined in (4) induces naturally an action $\Psi^{2,\sharp}$ of $\mathrm{Sp}(W)$ on $S^2(W)^*$:

$$\Psi^{2,\sharp} : \mathrm{Sp}(W) \times S^2(W)^* \to S^2(W)^* : (g, \underline{p}) \mapsto \Psi^{2,\sharp}_g \underline{p} := (\Psi^2_{g^{-1}})^* \underline{p} \ ,$$

i.e.,

$$\langle \Psi^{2,\sharp}_g \underline{p}, w \otimes w \rangle = \left\langle \underline{p}, (g^{-1} w) \otimes (g^{-1} w) \right\rangle \ . \qquad (8)$$

The symplectic group $\mathrm{Sp}(W)$ has a natural representation on $P^2(W)$:

$$\Pi^2 : \mathrm{Sp}(W) \times P^2(W) \to P^2(W) : (g, p) \mapsto \Pi^2_g p \ , \qquad (\Pi^2_g p)(w) := p(g^{-1} w) \ , \qquad (9)$$

which, according to (7) and (8), reads

$$(\Pi^2_g p)(w) = \frac{1}{2}\,\underline{p}(g^{-1}w, g^{-1}w)\ .$$

Clearly, these definitions make the representations of $\mathrm{Sp}(W)$ on $S^2(W)$ and $P^2(W)$ compatible with the isomorphism between them.

5.3.3 $\mathrm{Sp}(W)$-intertwining isomorphism $P : \mathrm{sp}(W) \to P^2(W)$

Every homogeneous quadratic polynomial $p \in P^2(W)$ can be written as

$$p(w) = \frac{1}{2}\,\omega(Bw, w)$$

for some matrix B since the symplectic matrix ω is nondegenerate. What conditions does the matrix B satisfy? Using the symmetry of $\underline{p}$ and antisymmetry of ω, we obtain

$$\frac{1}{2}\,\omega(Bw, w') = \frac{1}{2}\,\underline{p}(w, w') = \frac{1}{2}\,\underline{p}(w', w) = \frac{1}{2}\,\omega(Bw', w) = -\frac{1}{2}\,\omega(w, Bw')\ ,$$

that is,

$$\omega(Bw, w') + \omega(w, Bw') = 0\ ,$$

which is exactly the condition (3) on the elements of the Lie algebra $\mathrm{sp}(W)$. The constructed correspondence

$$P : \mathrm{sp}(W) \to P^2(W) : B \mapsto P(B)\ , \qquad P(B)(w) := \frac{1}{2}\,\omega(Bw, w) \tag{10}$$

intertwines the adjoint representation of $\mathrm{Sp}(W)$ on $\mathrm{sp}(W)$ and the representation Π^2 (9) of $\mathrm{Sp}(W)$ on $P^2(W)$:

$$\begin{aligned} P(\mathrm{Ad}_g B)(w) &= \frac{1}{2}\,\omega(\mathrm{Ad}_g B\, w, w) = \frac{1}{2}\,\omega(gBg^{-1}\, w, w) \\ &= \frac{1}{2}\,\omega(Bg^{-1}\, w, g^{-1}w) = P(B)(g^{-1}w) = \Pi^2_g(P(B))(w)\ . \end{aligned}$$

5.3.4 $\mathrm{Sp}(W)$

Now we construct the moment map

$$\Phi_W : W \to \mathrm{sp}(W)^*$$

of the action Ψ (2) of $\mathrm{Sp}(W)$ on W. The Hamiltonian vector field $\tilde{A} \in \mathcal{X}(W)$ corresponding to $A \in \mathrm{sp}(W)$ is defined as

$$\tilde{A}(w) = \frac{d}{dt}\bigg|_{t=0} \Psi_{\exp(tA)}\, w = \frac{d}{dt}\bigg|_{t=0} e^{tA}\, w = Aw \ .$$

The map

$$\hat{\Phi}_W : \mathrm{sp}(W) \to C^\infty(W) : A \mapsto \hat{\Phi}_W(A)$$

is constructed by requiring that

$$i_{\tilde{A}}\, \omega = d\big[\hat{\Phi}_W(A)\big] \ ,$$

i.e.,

$$\omega_{ij} A^i{}_k w^k = \frac{\partial}{\partial w^j}\big[\hat{\Phi}_W(A)(w)\big] \ ,$$

which implies that we can take

$$\hat{\Phi}_W(A)(w) = \frac{1}{2}\,\omega_{ij} A^i{}_k w^k w^j = \frac{1}{2}\,\omega(Aw, w) \ .$$

Therefore,

$$\hat{\Phi}_W(A) = P(A) \ ,$$

where $P(A)$ is defined in (10).

Now it is easy to construct the moment map

$$\Phi_W : W \to \mathrm{sp}(W)^* \ :$$

we have

$$\Phi_W(w)(A) := \hat{\Phi}_W(A)(w) = \frac{1}{2}\,\omega(Aw, w) \ .$$

5.4 Generalizing to $\mathrm{Sp}(V \otimes W)$

We perform a similar construction for the moment map

$$\Phi : V \otimes W \to (\mathrm{o}(V) \times \mathrm{sp}(W))^* \subset \mathrm{sp}(V \otimes W)^*$$

of the action of $\mathrm{O}(V) \times \mathrm{Sp}(W)$ on $V \otimes W$, following the strategy of [4]. The Hamiltonian vector field $\widehat{A \times B} \in \mathcal{X}(V \otimes W)$ corresponding to $A \times B \in (\mathrm{o}(V) \times \mathrm{sp}(W))$ is defined as

$$\left[\widehat{A \times B}\right](v \otimes w) = \left.\frac{d}{dt}\right|_{t=0} e^{t(A \times B)}\, v \otimes w = Av \otimes Bw \ .$$

The map

$$\begin{aligned} \hat{\Phi} : \mathrm{o}(V) \times \mathrm{sp}(W) &\to C^{\infty}(V \otimes W) \\ A \times B &\mapsto \hat{\Phi}(A \times B) \end{aligned}$$

is constructed by requiring that

$$d\big[\hat{\Phi}(A \times B)\big] = i\left(\widehat{A \times B}\right)\Omega \ ,$$

where Ω is the symplectic form on $V \otimes W$ defined in (1). In components and taken at a specific point, this reads

$$\left(d\hat{\Phi}_{A \times B}\right)_{v \otimes w}(v' \otimes w') = \left[g_{ij} A^i{}_k\, \omega_{\mu\nu} B^{\mu}{}_{\rho}\, (v \otimes w)^{k\rho}\, d(v \otimes w)^{j\nu}\right](v' \otimes w')$$

where (g_{ij}) is the metric (inner product) on V. Taking one coordinate of $d\Phi_{A \times B}$,

$$\frac{\partial \hat{\Phi}_{A \times B}}{\partial (v \otimes w)^{j\nu}}(v \otimes w) = g_{ij} A^i{}_k\, \omega_{\mu\nu} B^{\mu}{}_{\rho}\, (v \otimes w)^{k\rho}$$

Therefore, a valid choice for $\hat{\Phi}$ is

$$\left(\hat{\Phi}_{A \times B}\right)(v \otimes w) = \frac{1}{2}\, g(Av, v)\, \omega(Bw, w) \ ,$$

i.e.

$$(\Phi_{v \otimes w})\,(A \times B) = \frac{1}{2}\, g(Av, v)\, \omega(Bw, w) \tag{11}$$

We can project Φ to the moment maps Φ_W and Φ_V for the actions of the subgroups $\mathrm{Sp}(W)$ and $\mathrm{SO}(V)$. For $\mathrm{Sp}(W)$, set A to the identity in (11) to obtain Φ_W. For $\mathrm{SO}(V)$, first choose a symplectic basis for W:

$$e_1, ..., e_n, f_1, ..., f_n$$

and let $D : W \to W$ be the linear operator defined by

$$\begin{aligned} D\, e_j &= -f_j \\ D\, f_j &= e_j \ . \end{aligned} \tag{12}$$

Set B to D in (11) to obtain Φ_V.

5.5 Alternative expressions for the moment maps

It can easily be seen that

$$S^2(V \otimes W) \cong \Big[\Lambda^2(V) \otimes \Lambda^2(W)\Big] \oplus \Big[S^2(V) \otimes S^2(W)\Big] \ . \tag{13}$$

The symplectic form on W gives us a map

$$\begin{aligned} \Lambda^2(W) &\to \mathbb{R} \\ w_1 \wedge w_2 &\mapsto \omega(w_1, w_2) \end{aligned}$$

so that we can map the first summand in (13) to $\Lambda^2(V) \cong \mathrm{o}(V)^*$:

$$\begin{aligned} \sigma_V : \Lambda^2(V) \otimes \Lambda^2(W) &\to \Lambda^2(V) \cong \mathrm{o}(V)^* \\ (v_1 \wedge v_2) \otimes (w_1 \wedge w_2) &\mapsto \omega(w_1, w_2) v_1 \wedge v_2 \end{aligned}$$

Let proj_1 be the projection onto the first summand in (13), and let Φ'_V be the following map from $S^2(V \otimes W) \cong \Big[\Lambda^2(V) \otimes \Lambda^2(W)\Big] \oplus \Big[S^2(V) \otimes S^2(W)\Big]$ to $\mathrm{o}(V)^*$:

$$\begin{aligned} \Phi'_V : S^2(V \otimes W) &\to \mathrm{o}(V)^* \cong \Lambda^2\, V \\ v_1 \otimes w_1 \odot v_2 \otimes w_2 &\mapsto \frac{1}{2}\, g\Big((\cdot) v_1, v_2\Big) \omega\Big(D\, w_1, w_2\Big) \ , \end{aligned}$$

where D is the map defined in (12). This map may be expressed as

$$\Phi'_V = \sigma_V \circ \mathrm{proj}_1$$

where σ_V is understood to be acting on $(v_1 \wedge v_2) \otimes (Dw_1 \wedge w_2)$. Then our moment map Φ_V is the quadratic map from $V \otimes W$ to $\mathrm{o}(V)^*$ associated with Φ'_V.

We can perform an analogous construction for the moment map Φ_W of the group action of $\mathrm{Sp}(W)$ on $V \otimes W$. We use the inner product on V to generate a map

$$\begin{aligned} S^2(V) &\to \mathbb{R} \\ (v_1 \odot v_2) &\mapsto (v_1, v_2)_V \end{aligned}$$

where $v_1 \odot v_2$ denotes the symmetric tensor product of v_1 and v_2. So we can construct the map

$$\begin{aligned} \sigma_W : S^2(V) \otimes S^2(W) &\to S^2(W) \cong \mathrm{sp}(W)^* \\ (v_1 \odot v_2) \otimes (w_1 \odot w_2) &\mapsto g(v_1, v_2) w_1 \odot w_2 \end{aligned}$$

and define a function Φ'_W from $S^2(V \otimes W)$ to $\mathrm{sp}(W)^* \cong S^2(W)$:

$$\begin{aligned} \Phi'_W &= \sigma_W \circ \mathrm{proj}_2 \\ (v_1 \odot v_2) \otimes (w_1 \odot w_2) &\mapsto \frac{1}{2} g\Big(v_1, v_2\Big)\, \omega\Big((\cdot) w_1, w_2\Big) \end{aligned}$$

where proj_2 is the projection onto the second summand in (13). Then our moment map Φ_W is the quadratic map from $V \otimes W$ to $\mathrm{sp}(W)^*$ associated with Φ'_W.

5.6 $\mathcal{K}$ as a Marsden-Weinstein reduced space

We will now apply this general construction to the specific case $V = \mathbb{R}^6$ with the standard Euclidean inner product and $W = \mathbb{R}^2$ with the standard symplectic form. Taking the standard basis for $\mathbb{R}^2$, we introduce the following notation for elements of $V \otimes W$:

$$\begin{aligned} \begin{pmatrix} u \\ 0 \end{pmatrix} &:= u \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ v \end{pmatrix} &:= v \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Then the moment map

$$\Phi_V = \Phi_{\mathbb{R}^6} : \mathbb{R}^6 \otimes \mathbb{R}^2 \to \Lambda^2(\mathbb{R}^6) \cong \mathrm{o}(V)^*$$

is given by

$$\Phi_{\mathbb{R}^6}\begin{pmatrix} u \\ v \end{pmatrix}(C) = \frac{1}{2}[(Cu, v) - (Cv, u)] = (Cu, v)$$

where $C \in \mathrm{o}(V)$ and the last equality follows from the fact that C is by definition antisymmetric. Using our identification of $\mathrm{o}(V)^*$ with $\Lambda^2(V)$, we can rewrite this formula as

$$\Phi_{\mathbb{R}^6}\begin{pmatrix} u \\ v \end{pmatrix} = u \wedge v$$

Similarly, we apply our general construction and find that the moment map

$$\Phi_W = \Phi_{\mathbb{R}^2} : \mathbb{R}^6 \otimes \mathbb{R}^2 \to \mathrm{sp}(2, \mathbb{R}) \cong \mathrm{sp}(2, \mathbb{R})^*$$

is given by

$$\Phi_{\mathbb{R}^2}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \cdot v & \|v\|^2 \\ -\|u\|^2 & -u \cdot v \end{pmatrix}$$

We now consider $\Phi_{\mathbb{R}^2}^{-1}(0)$. Clearly it contains all pairs of vectors satisfying the defining conditions of $\mathcal{K}$:

$$\begin{aligned} &(u, v) = \|u\| = \|v\| = 0 \\ \Rightarrow &\Phi_{\mathbb{R}^2}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \cdot v & \|v\|^2 \\ -\|u\|^2 & -u \cdot v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

The stationary subgroup of 0 is equal to the whole $\mathrm{SL}(2, \mathbb{R}) \cong \mathrm{Sp}(2, \mathbb{R})$: a matrix $g \in \mathrm{SL}(2, \mathbb{R})$ acts as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto (g^T)^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} du - cv \\ -bu + av \end{pmatrix}$$

Expanding the bivector $(du - cv) \wedge (-bu + av)$,

$$\begin{aligned} (du - cv) \wedge (-bu + av) &= (du \wedge -bu) + (du \wedge av) + (-cv \wedge -bu) + (-cv \wedge av) \\ &= -db(u \wedge u) + ad(u \wedge v) - bc(u \wedge v) - ca(v \wedge v) \\ &= (ad - bc)(u \wedge v) \\ &= \det(g)(u \wedge v) \\ &= (u \wedge v) \end{aligned}$$

Thus, we see that $\Phi_{\mathbb{R}^6}^{-1}(u \wedge v)$ is an $\mathrm{SL}(2, \mathbb{R})$-invariant manifold. Let $O = \Phi_{\mathbb{R}^6}^{-1}(u \wedge v)$ be the preimage of a fixed element $u \wedge v$ with

$$u \wedge v \in \Phi_{\mathbb{R}^6}\left(\Phi_{\mathbb{R}^2}^{-1}(0)\right)$$

Then O is a whole $\mathrm{SL}(2, \mathbb{R})$-orbit: the group action of $\mathrm{SL}(2, \mathbb{R})$ on O is free and transitive. To see the transitivity, note that

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in O$$

means exactly that $u_1 \wedge v_1 = u_2 \wedge v_2$. This implies that u_1, and v_1 lie in the span of u_2 and v_2, i.e.

$$\exists\, a, b, c, d \in \mathbb{R} : u_1 = a u_2 + b v_2,\ v_1 = c u_2 + d v_2$$

We can again expand this bivector and show that the condition that the two wedge products are equal is exactly the condition that $ad - bc = 1$. This says precisely that given any two vectors in O, there is a matrix from $\mathrm{SL}(2, \mathbb{R})$ which maps one to the other, i.e. the group action is transitive.

Therefore, we may regard $\mathcal{K}$ as a Marsden-Weinstein reduced space:

$$\mathcal{K} \cong \Phi_{\mathbb{R}^2}^{-1}(0)/\mathrm{SL}(2, \mathbb{R}) \cong \Phi_{\mathbb{R}^2}^{-1}(0)/\mathrm{Sp}(2, \mathbb{R})$$

6 Appendix

6.1 Symplectic Manifolds

Let M be a finite-dimensional smooth manifold. A *symplectic structure* or *symplectic form* on M is a differential 2-form $\omega \in C^\infty(\Lambda^2 \tau^*(M))$ that satisfies the following:

$$d\omega = 0 \tag{14}$$

$$\forall m \in M : \forall X_m \in \tau_m(M) \backslash \{0\} : \exists Y_m \in \tau_m(M) : \omega_m(X_m, Y_m) \neq 0 \tag{15}$$

The pair (M, ω) is called a *symplectic manifold*.

Theorem (Darboux): Every symplectic manifold (M, ω) is of even dimension (dimension $2n$) and locally there exist coordinates $(q^1, ..., q^n, p^1, ..., p^n)$ such that

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp^i$$

That is, there exists a basis in which the matrix of ω is

$$(\omega) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

These coordinates are called *canonical*.

Let (M, ω) and (N, ω') be two symplectic manifolds. Then a transformation ϕ from $M \to N$ is called a *symplectomorphism* (or *canonical transformation* in physics parlance) if it is a diffeomorphism and satisfies

$$\phi^* \omega' = \omega$$

where $\phi*$ is the pullback of ϕ.

6.2 Symplectic Action of a Group

Let G be a Lie group and (M, ω) be a symplectic manifold. A group action of G on M

$$\begin{aligned} \phi : G \times M &\to M \\ (g, m) &\mapsto \phi_g(m) \end{aligned}$$

is called *symplectic* if

$$\forall g \in G : \phi_g \omega = \omega$$

That is, the transformation of M determined by any element of G is a symplectomorphism.

Let G be a Lie group acting on a symplectic manifold (M, ω) by symplectomorphisms, and let $\mathfrak{g}$ be the Lie algebra associated with G. Then every $g \in \mathfrak{g}$ defines a vector field $\tilde{\xi} \in C^\infty(\tau(M))$ as follows. Let $m \in M$ and $f \in C^\infty(M)$:

$$\tilde{\xi}(m)(f) := \frac{d}{dt}(f \circ \phi_{\exp t\xi}(m))\Big|_{t=0}$$

where $\phi_{\exp t\xi}$ is the one-parameter group of symplectomorphisms generated by ξ (for each $t \in \mathbb{R}$, $\exp t\xi$ is an element of G and thus defines a symplectomorphism according to the group action. Furthermore, these transformations form a group under composition). All these transformations are symplectomorphisms, meaning their pullbacks do not change ω, so for a vector field $\tilde{\xi}$ of type we have

$$0 = \frac{d}{dt}(\phi^*_{\exp t\xi}\omega)\Big|_{t=0} = L_{\tilde{\xi}}\omega = i(\tilde{\xi})d\omega + d(i(\tilde{\xi})\omega)$$

and because ω is closed, i.e. $d\omega = 0$, we obtain $d(i(\tilde{\xi})\omega) = 0$.
Any vector field $\tilde{\xi} \in C^\infty(\tau(M))$ which satisfies

$$d(i(\tilde{\xi})\omega) = 0$$

is called an *infinitessimal symplectic transformation* (or, in physics, an infinitesimal canonical transformation).

6.3 The Hamiltonian and Hamilton's Equations

Let $\tilde{\xi} \in C^\infty(\tau(M))$ be an infinitesimal symplectic transformation. Then there exists locally (or globally if $H^1(M) = 0$) a function $H \in C^\infty(M)$ (or rather, an equivalence class of functions, 2 functions being equivalent if they differ by an additive constant) defined by

$$i(\tilde{\xi})\omega = dH$$

This function is called the *Hamiltonian*.
In canonical coordinates (q^i, p^i), $i \in \{1, ..., n\}$,

$$\tilde{\xi} = \tilde{\xi}^i \frac{\partial}{\partial q^i} + \tilde{\xi} \frac{\partial}{\partial p^i}$$

so

$$\begin{aligned} i(\tilde{\xi})\omega &= -\tilde{\xi}^{n+i} dq^i + \tilde{\xi}^i dp^i \\ &=: dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p^i} dp^i \end{aligned}$$

and therefore

$$\tilde{\xi} = \frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i}$$

This says that the integral curves of the vector field $\tilde{\xi}$ are given by

$$\frac{d}{dt} \begin{pmatrix} q^i \\ p^i \end{pmatrix} = \tilde{\xi} = \begin{pmatrix} \frac{\partial H}{\partial p^i} \\ -\frac{\partial H}{\partial q^i} \end{pmatrix}$$

These equations are called *Hamilton's equations*.

6.4 The Poisson Bracket

Given a symplectic manifold (M, ω), we may define a Lie algebra structure on the vector space of smooth functions on M. The Lie bracket we will define on $C^\infty(M)$, called the *Poisson bracket*, obeys one additional axiom, and the resulting structure is called a *Poisson algebra*.

If H_1, $H_2 \in C^\infty(M)$, then their Poisson bracket

$$\begin{aligned} \{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) &\to C^\infty(M) \\ (H_1, H_2) &\mapsto \{H_1, H_2\} \end{aligned}$$

is defined by

$$\{H_1, H_2\} := -\tilde{\xi}_{H_1} H_2$$

where $\tilde{\xi}_{H_1} \in C^\infty(\tau(M))$ is the vector field corresponding to H_1. That is,

$$\tilde{\xi}_{H_1} = \frac{\partial H_1}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial H_i}{\partial q^i}\frac{\partial}{\partial p^i}$$

We can give several other expressions for $\{H_1, H_2\}$:

$$\begin{aligned} \{H_1, H_2\} &= -\tilde{\xi}_{H_1} H_2 \\ &= -dH_2(\tilde{\xi}_{H_1}) \\ &= -i(\tilde{\xi}_{H_1}) dH_2 \\ &= -i(\tilde{\xi}_{H_1}) i(\tilde{\xi}_{H_2})\omega \\ &= \omega(\tilde{\xi}_{H_1}, \tilde{\xi}_{H_2}) \\ &= \frac{\partial H_i}{\partial q^i}\frac{\partial H_2}{\partial p^i} - \frac{\partial H_1}{\partial p^i}\frac{\partial H_2}{\partial q^i} \end{aligned}$$

From this last expression especially, it is easy to see that the Poisson bracket is bilinear and antisymmetric, and also satisfies the derivative property (Leibniz rule):

$$\{H_1, H_2 H_3\} = \{H_1, H_2\}H_3 + H_2\{H_1, H_3\}$$

More generally, if A is a commutative ring then a Poisson bracket on A is a function

$$\begin{aligned} \{\cdot,\cdot\} : A \times A &\to A \\ (f_1, f_2) &\mapsto \{f_1, f_2\} \end{aligned}$$

with the following properties:

1. bilinearity
2. antisymmetry
3. derivative property/ Leibniz rule: $\{f_1, f_2f_3\} = \{f_1, f_2\}f_3 + f_2\{f_1, f_3\}$
4. Jacobi identity: $\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$

A Poisson algebra is a commutative ring with a Poisson bracket, i.e. a Lie bracket which also has the derivative property.

Let $Ham(M)$ be the real vector space of all vector fields $\tilde{\xi}$ such that $\exists H \in C^\infty(M) : i(\tilde{\xi})\omega = dH$. Our symplectic form ω induces a morphism of vector spaces

$$C^\infty(M) \to Ham(M)$$
$$H \mapsto \tilde{\xi}_H$$

Note that, if M is connected, $Ham(M) \cong C^\infty(M)/\mathbb{R}$ since $\tilde{\xi}_H$ determines H up to a constant. That is, if M is connected, the above map has a kernel consisting of the constant functions. Connectedness is required because if M had, say, 2 connected components then it could have a different constant value on each component while still having everywhere a differential of 0. In other words, we have the short exact sequence

$$0 \to \mathbb{R} \to C^\infty(M) \to Ham(M) \cong C^\infty(M)/\mathbb{R} \to 0$$

Using the identities $L_{\tilde{\xi}_h}\omega = 0$, $L = i \circ d + d \circ i$, and $d \circ dH = 0$, we obtain

$$\tilde{\xi}_{\{H_1,H_2\}} = -\left[\tilde{\xi}_{H_1}, \tilde{\xi}_{H_2}\right]$$

meaning that the maps in our short exact sequence are Lie algebra morphisms, with $\mathbb{R}$ equipped with identically zero Lie bracket. Only constant functions have zero Poisson bracket with every function, so the image of $\mathbb{R}$ under the map $\mathbb{R} \to C^\infty(M)$ is exactly the center of $C^\infty(M)$

6.5 The Moment Map

Let $\phi : G \times M \to M$ be the symplectic action of Lie group G on the symplectic manifold (M, ω). Let

$$Ad : G \times \mathfrak{g} \to \mathfrak{g}$$

be the adjoint representation of G on $\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, and let

$$\begin{aligned} Ad^\sharp : G \times \mathfrak{g}^* &\to \mathfrak{g}^* \\ (g, \mu) &\mapsto Ad^\sharp_g \mu := (Ad_{g^{-1}})^* \mu \end{aligned}$$

be the coadjoint representation.
The composition of the map

$$\begin{aligned} \tilde{}: \mathfrak{g} &\to Ham(M) \\ \xi &\mapsto \tilde{\xi} \end{aligned}$$

and the map $Ham(M) \to C^\infty(M)/\mathbb{R}$ (defined locally) gives a locally defined map

$$\begin{aligned} \hat{\Phi} : \mathfrak{g} &\to C^\infty(M) \\ \xi &\mapsto \hat{\Phi}(\xi) \end{aligned}$$

The symplectic action of G on (M, ω) is said to be *Hamiltonian* if there exists a Lie algebra morphism

$$\hat{\Phi} : \mathfrak{g} \to C^\infty(M)$$

which is also a G-morphism, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{sp}(W) & \xrightarrow{\hat{\Phi}} & C^\infty(M) \\ {\scriptstyle Ad_g}\downarrow & & \downarrow{\scriptstyle \phi_g^*} \\ \mathrm{sp}(W) & \xrightarrow{\hat{\Phi}} & C^\infty(M) \end{array}$$

and if it further satisfies

$$i(\tilde{\xi})\omega = d\hat{\Phi}(\xi)$$

It can be proven that if the action of G is Hamiltonian, then the map $\hat{\Phi}$ is globally, and not only locally, defined. Since $\hat{\Phi}$ is linear, if the action of our Lie group G on (M, ω) is Hamiltonian we obtain the map

$$\begin{aligned} \Phi : M &\to \mathfrak{g} \\ m &\mapsto \Phi(m) \end{aligned}$$

defined by

$$\Phi(m)(\xi) := \hat{\Phi}(\xi)(m)$$

That is,

$$\begin{aligned} d\Phi(m)(\xi) &= i(\tilde{\xi}(m))\omega \\ &= i\left(\frac{d}{dt}\phi_{\exp t\xi}(m)\Big|_{t=0}\right)\omega \end{aligned}$$

Φ is called the *moment map.* Because the action of G is Hamiltonian, Φ is equivariant:

$$\Phi \circ \phi_g = Ad^{\sharp}_g \circ \Phi$$

Theorem: Let $H : M \to \mathbb{R}$ be a G-invariant function, i.e. $H(\phi_g(m)) = H(m)$ for all $m \in M$ and all $g \in G$. Then $\hat{\Phi}(\xi_H) \in C^\infty(M)$ is an integral of the motion generated by H.

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