MATH 3413 Additional problems assigned on 1/28/14

Additional Problem 1. In this problem you will study the behavior of the solutions of autonomous ordinary differential equations, i.e., equations of the form

$$\frac{dx}{dt} = f(x) \ .$$

In each of the part (A) and (B) of this problem, you have to do the following:

(i) Find all equilibrium solutions of the ODE $\frac{dx}{dt} = f(x)$.

(ii) Sketch the graph of the function $f(x)$, and classify the equilibrium solutions you found in part (i). Put arrows to indicate the direction of the change of x with time.

(iii) In the (t, x)-plane, draw the equilibrium solutions of the ODE and sketch several other solutions to show roughly their behavior.

(iv) Solve the ODE explicitly (i.e., find its general solution).

The following trick will be useful (this is a particular case of the so-called *partial fraction decomposition* – see page 465 of the book): if you need to integrate an expression of the form $\frac{1}{(x-a)(x-b)}$ with $a \neq b$, you can find constants A and B such that

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \ ;$$

the integral of the right-hand side is standard.

(A) $\displaystyle \frac{dx}{dt} = 5 - x \, ;$

(B) $\displaystyle \frac{dx}{dt} = x(3 - x) \, .$

Additional Problem 2. Consider the autonomous differential euqation

$$\frac{dx}{dt} = \mu x - x^3 \ , \qquad (1)$$

where μ is a parameter. Let $f(x) := \mu x - x^3$ be the right-hand side of (1).

(a) If $\mu \leq 0$, show that the only equilibrium solution of the ODE (1) is $x(t) \equiv 0$, and it is stable. Sketch the graph of $f(x)$ and indicate how you came to your conclusion.

Hint: Computing $f'(x)$ might help you draw conclusions about the behavior of $f(x)$.

(b) If $\mu > 0$, show that the equilibrium $x^*(t) \equiv 0$ of the ODE (1) is now unstable, but there are two new equilibria, $x_1^* = -\sqrt{\mu}$ and $x_2^* = -\sqrt{\mu}$, which are stable. Again, sketch the graph of $f(x)$ for $\mu > 0$, and show on it how x changes with time.

(c) From your findings in parts (a) and (b), you can conclude that the qualitative nature of the solutions of the ODE (1) changes dramatically at $\mu = 0$ as the parameter μ increases – such a value of μ is called a *bifurcation point* for the ODE (1). In the (μ, x)-plane (i.e., μ is on the horizontal axis, and x is on the vertical axis), plot the positions of the equilibrium solutions as functions of the parameter μ, for all values of μ – you will obtain a straight horizontal line that at some point splits into three branches. In your plot, denote the positions of the stable equilibria with a solid line, and the positions of the unstable equilibria with a dashed line.