MATH 3413 Problems assigned on 3/27/14

Remark: The definitions of a vector space (linear space) and inner product vector space are given at the end of this homework.

Problem 1. Recall that a *basis* in a vector space V is an (ordered) set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$ such that every vector $\mathbf{u} \in V$ can be written in a unique way in the form

$$\mathbf{u} = u_1\mathbf{v}_1 + u_2\mathbf{v}_2 + \cdots + u_k\mathbf{v}_k = \sum_{j=1}^{k} u_j\mathbf{v}_j \ .$$

The numbers u_j are called the *components* of the vector $\mathbf{u}$ in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$. It can be shown that each basis of a vector space V contains the same number of vectors; the number of vectors in a basis of V is called the *dimension* of V and is denoted by $\dim V$.

As discussed in class, the set of all polynomials of degree no greater than n form a vector space, which we denote by V_n. An element $p \in V_n$ is a polynomial

$$p(x) = p_0 + p_1x + \cdots + p_{n-1}x^{n-1} + p_nx^n \ . \tag{1}$$

If p (defined above) and q, defined by

$$q(x) = q_0 + q_1x + \cdots + q_{n-1}x^{n-1} + q_nx^n \ ,$$

are two polynomials from V_n, and α is a real number, the sum $p + q \in V_2$ of p and q and the product $\alpha p \in V_2$ are defined by

$$\begin{aligned} (p+q)(x) &:= (p_0+q_0) + (p_1+q_1)x + \cdots + (p_{n-1}+q_{n-1})x^{n-1} + (p_n+q_n)x^n \ , \\ (\alpha p)(x) &:= \alpha p_0 + \alpha p_1x + \cdots + \alpha p_{n-1}x^{n-1} + \alpha p_nx^n \ . \end{aligned} \tag{2}$$

If we define the polynomials e_j by

$$e_j(x) = x^j \ , \qquad j = 0, 1, 2, \ldots \ ,$$

then it is clear that the polynomials $\{e_0, e_1, \cdots, e_{n-1}, e_n\}$ form a basis of V_n, in which the polynomial $p \in V_n$ defined in (1) has components p_0, p_1, …, p_{n-1}, p_n. Clearly, $\dim V_n = n+1$.

(a) Find the components of the quadratic polynomial

$$p = p_0e_0 + p_1e_1 + p_2e_2 = \sum_{j=0}^{2} p_je_j \in V_2 \ ,$$

that is,

$$p(x) = p_0 + p_1x + p_2x^2 \ , \tag{3}$$

in the basis $\{f_0, f_1, f_2\}$ of V_2 defined by

$$f_0 = \tfrac{1}{5}e_0 \ , \quad f_1 = 3e_1 - 2e_2 \ , \qquad f_2 = e_0 - 2e_1 + 3e_2 \ . \tag{4}$$

Please write your calculations in detail.

(b) Explain in a couple of sentences why your result from part (a) implies that the set of polynomials $\{f_0, f_1, f_2\}$ defined by (4) indeed is a basis of V_2.

(c) Demonstrate that the set of vectors

$$g_0 = 5e_0 \ , \quad g_1 = 2e_0 + 3e_1 - 4e_2 \ , \qquad g_2 = e_0 + 3e_1 - 4e_2$$

is *not* a basis of V_2.

Hint: You can do this by finding a vector $h \in V_2$ that can be expressed as a linear combination of the vectors g_0, g_1, and g_2 in more than one way.

Problem 2. Seth defined a family of polynomials, which he modestly denoted by s_0, s_1, s_2, ..., that satisfy the following conditions:

(i) the polynomial s_k is of degree k;

(ii) the polynomials s_k are *monic*, i.e., the coefficient in front of the term with the highest power of x in s_k (in our case, this is the coefficient of x^k) is equal to 1;

(iii) the polynomials s_0, s_1, s_2, ..., s_n form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = \mathrm{e}^{-x})$ (defined below).

In condition (iii) above, $V_n(a, b; w(x))$ stands for the linear space of polynomials of degree no greater than n endowed with the inner product

$$\langle p, q \rangle = \int_a^b p(x)\, q(x)\, w(x)\, \mathrm{d}x \ ,$$

where w is a non-negative function that is allowed to take value zero only at a set of isolated points. The inner product in the vector space $V_n(0, \infty; w(x) = \mathrm{e}^{-x})$ considered by Seth is, therefore,

$$\langle p, q \rangle = \int_0^\infty p(x)\, q(x)\, \mathrm{e}^{-x}\, \mathrm{d}x \ .$$

In the solution of this problem the following identity will be handy (where $0! := 1$):

$$\int_0^\infty x^k\, \mathrm{e}^{-x}\, \mathrm{d}x = k! \ , \qquad k = 0, 1, 2, \dots \ .$$

(a) Clearly, $s_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial s_1 of degree 1 that is orthogonal to s_0. In other words, find the only polynomial $s_1(x) = x + \alpha$ such that $\langle s_1, s_0 \rangle = 0$ (notice that the coefficient in front of x in s_1 is equal to 1 because of the requirement that the polynomial be monic).

(b) Find the only monic quadratic polynomial s_2 that is orthogonal to both s_0 and s_1.

(c) Show that the polynomial $p(x) = x^2 + 3$ can be represented as a linear combination of the polynomials s_0, s_1 and s_2 as follows: $p = s_2 + 4s_1 + 5s_0$.

(d) Show directly that $\langle s_0, s_0\rangle = 1$, $\langle s_1, s_1\rangle = 1$, $\langle s_2, s_2\rangle = 4$.

(e) The *angle* θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\langle \mathbf{u}, \mathbf{v}\rangle = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \ ,$$

where

$$\|\mathbf{u}\| := \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}$$

is the *norm* of the vector $\mathbf{u}$. Find the angle between the polynomials p (defined in part (c)) and s_1 (defined in part (a)).

Hint: This can be done with *very simple* calculations if you use the fact that the polynomials s_0, s_1, and s_2 are orthogonal to each other, and that p can be expressed as their linear combination as in part (c).

(f) Find the orthogonal projection, $\mathrm{proj}_{s_0+2s_1} p$, of the polynomial $p(x) = x^2 + 3$ onto the "straight line"

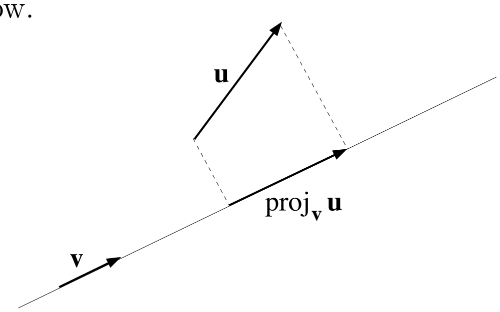
$$\ell := \{t(s_0 + 2s_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; \mathrm{e}^{-x})$. If you have solved part (c), then finding this orthogonal projection should be easy.

Hint: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the inner product linear space V, then the orthogonal projection of the vector $\mathbf{u}$ onto the straight line in the direction of $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\,\mathbf{v}$$

– see the picture below.



(g) The vectors s_0, s_1, and s_2 form a basis of the vector space $V_2(0, \infty; \mathrm{e}^{-x})$. By construction, this basis is *orthogonal*, i.e., $\langle s_i, s_j\rangle = 0$ if $i \neq j$. Use this fact and your results above to construct an *orthonormal* basis $\{\widetilde{s}_0, \widetilde{s}_1, \widetilde{s}_2\}$ of $V_2(0, \infty; \mathrm{e}^{-x})$, where $\widetilde{s}_j := \mu_j s_j$, for some positive constants $\mu_j > 0$ (depending on j) such that

$$\langle \widetilde{s}_i, \widetilde{s}_j\rangle = \delta_{ij} \ .$$

Here δ_{ij} is Kroneker's symbol, defined by $\delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

Problem 3. Find the general solutions of the following partial differential equations. Do not forget that they contain arbitrary functions; write explicitly the arguments of these functions.

(a) $u_x = y \sin x$, where $u = u(x, y, z)$.

(b) $u_{xy} = 0$, where $u = u(x, y)$.

(c) $u_{xx} = 3y^2$, where $u = u(x, y)$.

Definition 1. *A* vector space *(or* linear space*) is a set* $V = \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \dots\}$ *in which the following two operations are defined:*

(A) Addition of vectors: $\mathbf{u} + \mathbf{v} \in V$*, which satisfies the properties*

- *(A_1) associativity:* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$*;*
- *(A_2) existence of a zero vector: there exists a vector* $\mathbf{0} \in V$ *such that* $\mathbf{u} + \mathbf{0} = \mathbf{u}$ $\forall \mathbf{u} \in V$*;*
- *(A_3) existence of an opposite element:* $\forall\, \mathbf{u} \in V$ *there exists a vector* $\widetilde{\mathbf{u}} \in V$ *such that* $\mathbf{u} + \widetilde{\mathbf{u}} = \mathbf{0}$*;*
- *(A_4) commutativity:* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad \forall\, \mathbf{u}, \mathbf{v} \in V$*;*

(M) Multiplication of a scalar (i.e., a number) and a vector: $\alpha\mathbf{u} \in V$ *for* $\alpha \in \mathbb{R}$*, which satisfies the properties*

- *(M_1) distributivity w.r.t. addition of vectors:* $\alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u}+\alpha\mathbf{v} \quad \forall\, \alpha \in \mathbb{R}, \forall\, \mathbf{u}, \mathbf{v} \in V$*;*
- *(M_2) distributivity w.r.t. addition of scalars:* $(\alpha+\beta)\mathbf{u} = \alpha\mathbf{u}+\beta\mathbf{u} \quad \forall\, \alpha, \beta \in \mathbb{R}, \forall\, \mathbf{u} \in V$*;*
- *(M_3) distributivity w.r.t. multiplication:* $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u}) \quad \forall\, \alpha, \beta \in \mathbb{R}, \forall\, \mathbf{u} \in V$*;*
- *(M_4) normalization:* $1\mathbf{u} = \mathbf{u} \quad \forall\, \mathbf{u} \in V$*.*

Definition 2. *An* inner product vector space *is a vector space* V *with an operation* $\langle \sqcup, \sqcup \rangle$ *(where* $\sqcup$ *stands for a "spaceholder," i.e., for a slot for an argument) that satisfies the properties*

- *(I_1)* $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \quad \forall\, \mathbf{u}, \mathbf{v} \in V$*;*
- *(I_2)* $\langle \mathbf{u} + \alpha\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \alpha\langle \mathbf{v}, \mathbf{w} \rangle \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \forall\, \alpha \in \mathbb{R}$*;*
- *(I_3)* $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \quad \forall\, \mathbf{u} \in V$*; moreover,* $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ *only if* $\mathbf{u} = \mathbf{0}$.