MATH 3413 Problems assigned on 4/17/14

Problem 1. As we showed in class, the solution of the initial boundary value problem with zero Neumann boundary conditions

$$\begin{aligned} &u_{tt}(x,t) = 3^2 u_{xx}(x,t)\ , \qquad x \in [0,\pi]\ , \quad t > 0\ , \\ &\left.\begin{aligned} u_x(0,t) &= 0 \\ u_x(\pi,t) &= 0 \end{aligned}\right\} t > 0 \\ &\left.\begin{aligned} u(x,0) &= 7 \\ u_t(x,0) &= 30\cos 5x \end{aligned}\right\} x \in [0,\pi] \end{aligned} \tag{1}$$

has the form

$$u(x,t) = A_0 + B_0 t + \sum_{k=1}^{\infty} \left(A_k \cos ckt + B_k \sin ckt\right) \cos kx\ ,$$

where $c = 3$ is the speed of the wave in the string. Substitute this expression in (1) to find the values of all coefficients A_j and B_j (for $j = 0, 1, 2, \ldots$). Write down the solution of the initial boundary value problem (1).

Problem 2. The motion of a string with linear density ρ and tension τ that is subjected to an external force of linear density $F(x,t)$ and an air resistance force $-\Gamma u_t(x,t)$ is described by the PDE

$$\rho u_{tt}(x,t) = \tau u_{xx}(x,t) - \Gamma u_t(x,t) + F(x,t)\ .$$

Dividing by ρ and setting $c := \sqrt{\tau/\rho}$, $\gamma := \Gamma/\rho$, and $f(x,t) := F(x,t)/\rho$, we rewrite this equation in the form

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) - \gamma\, u_t(x,t) + f(x,t)\ .$$

Assume that the string has length L and is attached at both ends. Then the motion of the string is governed by the initial boundary value problem

$$\begin{aligned} &u_{tt}(x,t) = c^2 u_{xx}(x,t) - \gamma\, u_t(x,t) + f(x,t)\ , \qquad x \in [0,L]\ , \quad t > 0\ , \\ &\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} t > 0 \\ &\left.\begin{aligned} u(x,0) &= g(x) \\ u_t(x,0) &= h(x) \end{aligned}\right\} x \in [0,L] \end{aligned} \tag{2}$$

with $g(x)$ and $h(x)$ being the initial position and the initial velocity, respectively.

For simplicity, let us ignore the air resistance force, assume that the string has length π, that the speed of the waves in the string is $c = 3$, the external forcing term has the form

$f(x) = 5\sin 7x$ (in particular, notice that it does not depend on t), and that the initial position and velocity are both equal to zero. Then the problem (2) becomes

$$
\begin{aligned}
& u_{tt}(x,t) = 3^2 u_{xx}(x,t) + 5\sin 7x \ , \qquad x \in [0,\pi] \ , \quad t > 0 \ , \\
& \left.\begin{aligned} u(0,t) &= 0 \\ u(\pi,t) &= 0 \end{aligned}\right\} t > 0 \\
& \left.\begin{aligned} u(x,0) &= 0 \\ u_t(x,0) &= 0 \end{aligned}\right\} x \in [0,\pi]
\end{aligned}
\tag{3}
$$

Look for solution of the initial boundary value problem (3) in the form

$$
u(x,t) = \sum_{k=0}^{\infty} T_k(t) \sin kx \ ,
$$

where $c = 3$ is the speed of the waves in the string, and A_k and B_k are coefficients to be determined. Write down the ODEs for the functions $T_k(t)$ and the initial conditions for them and solve them all. There will be only one function $T_k(t)$ that will be non-zero. Write down the solution of the initial boundary value problem (3).

Problem 3. Consider the wave equation on the interval $[0,L]$ with zero Neumann boundary conditions:

$$
\begin{aligned}
& u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \ , \qquad x \in [0,L] \ , \quad t > 0 \ , \\
& \left.\begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0 \end{aligned}\right\} t > 0 \\
& \left.\begin{aligned} u(x,0) &= g(x) \\ u_t(x,0) &= h(x) \end{aligned}\right\} x \in [0,L]
\end{aligned}
\tag{4}
$$

The speed of the wave is equal to

$$
c = \sqrt{\frac{\tau}{\rho}} \ , \tag{5}
$$

where τ is the tension in the string (measured in Newtons), and ρ is the linear density of the string (measured in kg/m).

One can show that at time t the energy of the string is equal to

$$
E(t) = \int_0^L \left[\frac{\rho}{2} u_t(x,t)^2 + \frac{\tau}{2} u_x(x,t)^2 \right] \mathrm{d}x \ . \tag{6}
$$

In this expression, the term with $u_t(x,t)^2$ represents the kinetic energy of the string (recall that $u_t(x,t)$ is the velocity of the point of the string with coordinate x at time t and that the kinetic energy of a part of the string with length $\mathrm{d}x$ is $\frac{\mathrm{d}m}{2}\, u_t(x,t)^2 = \frac{\rho}{2} u_t(x,t)^2 \, \mathrm{d}x$), while the term with $u_x(x,t)^2$ is the potential energy of the string.

(a) Differentiate (6) with respect to time to find $E'(t)$; you can interchange $\frac{\mathrm{d}}{\mathrm{d}t}$ with the integration over x in the right-hand side of (6).

(b) Apply the PDE from (4) to the expression for $E'(t)$ obtained in part (a) and use (5) to show that the rate of change of the energy of the string is equal to

$$E'(t) = \tau \int_0^L (u_t\, u_{xx} + u_x\, u_{xt})\, \mathrm{d}x\ . \tag{7}$$

(c) Show that the integrand in the right-hand side of (7) can be written in the form $\frac{\mathrm{d}}{\mathrm{d}x}(u_x u_t)$. What property of differentiation have you used?

(d) Use your results from parts (b) and (c) to show that the rate of change of energy of the string is equal to

$$E'(t) = \tau \left[u_t(L,t)\, u_x(L,t) - u_t(0,t)\, u_x(0,t)\right]\ .$$

What mathematical result have you used in your derivation?

(e) Finally, use (4) to show that the energy of the vibration of the string is conserved (i.e., that it does not depend on time).