MATH 2433 Additional Problem assigned on 8/21/2014

As we discussed in class, vectors are objects for which two operations are defined: addition of two vectors, and multiplication of a vector by a scalar, and these two operations satisfy certain properties (which we take as axioms that define vectors). Below, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arbitrary vectors, and α and β are arbitrary scalars.

(+) Axioms of addition:

- ($+_1$) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity of addition);
- ($+_2$) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (associativity of addition);
- ($+_3$) there exists a vector $\mathbf{0}$ (called a *zero vector*) such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for any vector $\mathbf{a}$;
- ($+_4$) for any vector $\mathbf{a}$ there exists a vector $\widetilde{\mathbf{a}}$ such that $\mathbf{a} + \widetilde{\mathbf{a}} = \mathbf{0}$.

($\cdot$) Axioms of multiplication of a vector and a scalar:

- ($\cdot_1$) $(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a})$;
- ($\cdot_2$) $1\mathbf{a} = \mathbf{a}$.

($+\cdot$) Axioms connecting the addition of vectors with the multiplication of a vector and a scalar ("distributive laws"):

- ($+\cdot_1$) $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$;
- ($+\cdot_2$) $\alpha(\mathbf{a} + \beta) = \alpha\mathbf{a} + \alpha\mathbf{b}$.

In class we proved the following result (the symbol ■ indicates the end of the proof).

Theorem A. *The zero vector is unique.*

Proof. Assume that there are two zero vectors, $\mathbf{0}$ and $\mathbf{0}'$. Then we have

$$\mathbf{0} \overset{(1)}{=} \mathbf{0} + \mathbf{0}' \overset{(2)}{=} \mathbf{0}' + \mathbf{0} \overset{(3)}{=} \mathbf{0}' ,$$

Here we used the following facts (the numbering below corresponds to the numbering of the three equalities in the chain above):

- the equality (1) used the fact that $\mathbf{0}'$ is a zero vector, therefore $\mathbf{a} + \mathbf{0}' = \mathbf{a}$ for any vector $\mathbf{a}$, in particular, for $\mathbf{a} = \mathbf{0}$; in other words, we used Axiom ($+_3$) with $\mathbf{a} = \mathbf{0}$;
- the equality (2) used Axiom ($+_1$) (the commutativity of addition);
- the equality (3) used the fact that $\mathbf{0}$ is a zero vector, therefore $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for any vector $\mathbf{a}$, in particular, for $\mathbf{a} = \mathbf{0}'$; in other words, we used Axiom ($+_3$) with $\mathbf{a} = \mathbf{0}'$. ■

Problem 1. Complete the proof of the following theorem.

Theorem B. *Any vector* $\mathbf{a}$ *has only one "opposite" vector* $\widetilde{\mathbf{a}}$ *(which satisfies* $\mathbf{a}+\widetilde{\mathbf{a}}=\mathbf{0}$*).*

Proof. Assume that there are two vectors, $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{a}}'$, that are opposite to $\mathbf{a}$, i.e., such that $\mathbf{a}+\widetilde{\mathbf{a}}=\mathbf{0}$ and $\mathbf{a}+\widetilde{\mathbf{a}}'=\mathbf{0}$. Then the following equalities hold:

$$\widetilde{\mathbf{a}} \stackrel{(1)}{=} \widetilde{\mathbf{a}}+\mathbf{0} \stackrel{(2)}{=} \widetilde{\mathbf{a}}+(\mathbf{a}+\widetilde{\mathbf{a}}') \stackrel{(3)}{=} (\widetilde{\mathbf{a}}+\mathbf{a})+\widetilde{\mathbf{a}}' \stackrel{(4)}{=} \mathbf{0}+\widetilde{\mathbf{a}}' \stackrel{(5)}{=} \widetilde{\mathbf{a}}'+\mathbf{0} \stackrel{(6)}{=} \widetilde{\mathbf{a}}' .$$

The reasons why these equalities hold are the following:

- the equality (1) used Axiom $(+_3)$ (existence of a zero vector);
- the equality (2) used …;
- the equality (3) used …;
- the equality (4) used …;
- the equality (5) used …;
- the equality (6) used …. ■

Food for thought.[1] Think about the proofs of the following theorems.

Theorem C. *The product of* 0 *and any vector* $\mathbf{a}$ *is equal to the zero vector:* $0\mathbf{a}=\mathbf{0}$.

Proof. From Theorem A we know that the zero vector is unique. Therefore, if we know that for some vector $\mathbf{b}$ we have $\mathbf{a}+\mathbf{b}=\mathbf{a}$ for any vector $\mathbf{a}$, then $\mathbf{b}=\mathbf{0}$. Thus, if we prove that $\mathbf{a}+0\mathbf{a}=\mathbf{a}$ for an arbitrary vector $\mathbf{a}$, this would imply that $0\mathbf{a}=\mathbf{0}$. Indeed, we have

$$\mathbf{a}+0\mathbf{a} \stackrel{(1)}{=} 1\mathbf{a}+0\mathbf{a} \stackrel{(2)}{=} (1+0)\mathbf{a} \stackrel{(3)}{=} 1\mathbf{a} \stackrel{(4)}{=} \mathbf{a} ,$$

where the equality (1) holds because… ■

Theorem D. *For any vector* $\mathbf{a}$*, its opposite,* $\widetilde{\mathbf{a}}$*, is equal to* $(-1)\mathbf{a}$*.*

Proof. According to Theorem B, for any $\mathbf{a}$, the vector $\widetilde{\mathbf{a}}$ is unique. Therefore, if we prove that $\mathbf{a}+(-1)\mathbf{a}=\mathbf{0}$, this will imply that $\widetilde{\mathbf{a}}=(-1)\mathbf{a}$. The proof follows from the following chain of equalities: … ■

Theorem E. *For any vector* $\mathbf{a}$ *we have* $\mathbf{a}+\mathbf{a}=2\mathbf{a}$*.*

Proof. Do it yourself. ■

[1] "Food for thought" problems are interesting problems you may try to solve, but are not to be turned in.