MATH 3113 – Homework assigned on 9/6/13

In all problems of this homework assume that the arguments of the logarithms are positive. Of course, in real life one needs to be more careful!

Problem 1. Find the general solutions the following differential equations. Besides the general solutions, please find their singular solutions (if they have singular solutions).

(a) $y'' = 2y(y')^3$;

(b) $y'' = 2yy'$.

Problem 2. The following equation is called a *Riccati equation*:

$$\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x) \ . \tag{1}$$

(a) Suppose that one particular solution $y_1(x)$ of (1) is known. Show that the substitution

$$y(x) = y_1(x) + \frac{1}{v(x)}$$

transforms the Riccati equation into the linear equation

$$\frac{dv}{dx} + [B(x) + 2A(x)y_1(x)]\, v = -A(x) \ .$$

(b) Use the method from part (a) to solve the equation

$$\frac{dy}{dx} + 2xy = 1 + x^2 + y^2 \ .$$

Problem 3. Birth and death rates of animal populations typically are not constant; instead, they vary periodically with the passage of seasons. Find the population $P(t)$ if it satisfies the following initial-value problem (note that the ODE is non-autonomous)

$$\frac{dP}{dt} = [\alpha + \beta\cos(2\pi t)]\ P \ , \qquad P(0) = P_0 \ .$$

Here the time t is measured in years, and α and β are positive constants. You can think of the function $\mu(t) = \alpha + \beta\cos(2\pi t)$ (multiplying P in the right-hand side of the equation) as a periodically varying growth rate around its mean value α.

Please turn the page!

Problem 4. In this problem you will study the behavior of the solutions of autonomous ordinary differential equations of the form

$$\frac{dx}{dt} = f(x) \ .$$

In each of the part (A)–(C) of this problem, you have to do the following:

(i) Find all equilibrium solutions of the ODE $\frac{dx}{dt} = f(x)$.

(ii) Sketch the graph of the function $f(x)$, and classify the equilibrium solutions you found in part (i). Put arrows to indicate the direction of the change of x with time.

(iii) In the (t, x)-plane, draw the equilibrium solutions of the ODE and sketch several other solutions to show roughly their behavior.

(iv) Solve the ODE explicitly.

The following trick will be useful for some of the problems (this is a particular case of the so-called *partial fraction decomposition* – see page 465 of the book): if you need to integrate an expression of the form $\frac{1}{(x-a)(x-b)}$ with $a \neq b$, you can find constants A and B such that

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \ ;$$

the integral of the right-hand side is standard.

(A) $\displaystyle \frac{dx}{dt} = 5 - x \,;$

(B) $\displaystyle \frac{dx}{dt} = x(3 - x) \,;$

(C) $\displaystyle \frac{dx}{dt} = -x^2 + 5x - 4 \,.$