MATH 3113 – Homework assigned on 9/23/13

Sec. 3.3: Problems 2, 3, 5, 10, 26, 39.

Additional problem.

(a) Suppose that one solution $y_1(x)$ of the homogeneous second-order linear ODE

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = 0 \tag{1}$$

is known (on an interval I where the functions p, q, and r are continuous and p does not become zero). The method of *reduction of order* consists of substituting

$$y_2(x) = v(x)\,y_1(x)$$

in the ODE (1) and attempting to determine the function $v(x)$ so that $y_2(x)$ is a second linearly independent solution of (1). After substituting $y_2(x) = v(x)\,y_1(x)$ in (1), use the fact that $y_1(x)$ is a solution of (1), (i.e., that $p(x)\,y_1'' + q(x)\,y_1' + r(x)\,y_1 = 0$) to deduce that the function $v(x)$ satisfies the ODE

$$py_1(x)\,v'' + [2p(x)y_1'(x) + q(x)y_1(x)]\,v' = 0\ . \tag{2}$$

If $y_1(x)$ is known, then (2) is a separable equation of second order which does *not* contain the unknown function, so it can be reduced to a first-order equation by using the methods from Section 1.6 of the book.

(b) Use the method of reduction of order to find a second linearly independent solution, $y_2(x)$, of the homogeneous second-order linear ODE

$$(x+1)\,y'' - (x+2)\,y' + y = 0 \tag{3}$$

on the interval $I = (-1, \infty)$, if you know that one solution of this equation is the function $y_1(x) = e^x$.

Remark: Note that when you solve the ODE (2) for the function $v(x)$, you will obtain a general solution of (2) (i.e., a two-parameter family of solutions). Therefore, you have a lot of freedom in choosing the constants in this general solution to construct the solution $y_2(x)$. Choose some values of the constants that ensure that the new solution, $y_2(x)$, and the old one, $y_1(x)$, are linearly independent. Here is a *hint:* At some point, you will obtain that $v(x) = A(2+x)e^{-x} + B$, where A and B are arbitrary constants.

(c) Solve the initial value problem consisting of the ODE (3) and the initial conditions $y(0) = 3$, $y'(0) = 4$.

Remark: Note that, although you had freedom in choosing the second linearly independent solution $y_2(x)$ in part (b), the solution of the initial value problem in part (c) is determined uniquely!