MATH 2433 – Additional problem assigned on 10/9/14

Additional problem.

Let a and b be positive constants, and consider the vector function

$$\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}\ , \qquad t \in \mathbb{R}\ , \tag{1}$$

describing a helix in $\mathbb{R}^3$. One can easily see that this helix lies completely in the surface $x^2 + y^2 = a^2$, which is a vertical cylinder of radius a. It is also obvious that when t increases by 2π, the projection $\langle a\cos t, a\sin t\rangle$ of $\mathbf{r}(t)$ onto the (x, y)-plane traverses completely a circle of radius a while the height (i.e., the z-coordinate) of the point $\mathbf{r}(t)$ increases by $2\pi b$.

(a) Compute the tangent vector $\mathbf{r}'(t)$ to the curve $\mathbf{r}(t)$, and its magnitude $|\mathbf{r}'(t)|$.

(b) Find the unit tangent vector $\mathbf{T}(t)$ to the curve $\mathbf{r}(t)$.

(c) Find the curvature $\kappa = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|$ of the helix at the point $\mathbf{r}(t)$ by using that

$$\kappa(t) = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| = \left|\frac{\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}}{\frac{\mathrm{d}s}{\mathrm{d}t}}\right| = \frac{\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right|}{\left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}\ . \tag{2}$$

In other words, there is no need to reparameterize the helix by using the arc length as parameter – simply use the expressions for $\mathbf{T}(t)$ and $|\mathbf{r}'(t)|$ found in the previous parts of this problem. In the expression (2) for the curvature as the ratio of $|\mathbf{T}'(t)|$ and $|\mathbf{r}'(t)|$, the prime denotes differentiation with respect to t.

(d) The inverse of the curvature $\kappa(t)$ of the helix at the point $\mathbf{r}(t)$ is called the *radius of curvature* of the space curve at $\mathbf{r}(t)$:

$$R(t) := \frac{1}{\kappa(t)} \tag{3}$$

(it is easy to check that $R(t)$ defined by (3) indeed has units of length, so the name "radius" for $R(t)$ is consistent with the unit in which $R(t)$ is measured). The radius of curvature $R(t)$ has a simple geometric meaning – this is the radius of the "best fitting" circle to the curve at the point $\mathbf{r}(t)$.

Geometrically, what do you expect the radius of curvature of the helix to tend to if you take the limit $b \to 0$ while keeping a constant? Give a brief but convincing explanation without doing any calculations. (Hint: When $b \to 0$, the helix becomes a simpler curve whose radius of curvature you know.) Finally, check that the expression for $R(t)$ tends to the limit you predicted based on your geometric understanding of the problem.

(e) Geometrically, what do you expect the radius of curvature of the helix to tend to if the helix becomes infinitely elongated in z-direction, i.e., if you take the limit $b \to \infty$ while keeping a constant (or, equivalently, if you take the limit $a \to 0$ while keeping b constant). Again, you have to answer this question based on the geometry of the problem, and then to confirm your prediction by taking the limit in the expression for $R(t)$.