MATH 2433 – Additional problem assigned on 10/23/14

Additional problem.

Continued fractions are infinite fractions of the form

$$\cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4 + \cdots}}}} \; ,$$

where q_n are natural numbers (i.e., positive integers). One can find the value of certain continued fractions by using facts about sequences. In this problem you will find the value of the continued fraction

$$\sigma := \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}} \; , \tag{1}$$

which is sometimes called the *golden ratio* (sometimes "golden ratio" is used for the number $\phi := 1 + \sigma$). Its value can be easily found on a calculator to be $\sigma = 0.6180339887\ldots$. In this problem you will find the exact value of the golden ratio.

Consider the sequence defined recursively by

$$a_1 = 1 \; , \qquad a_{n+1} = \frac{1}{1 + a_n} \quad \text{for } n = 1, 2, 3, \ldots \; . \tag{2}$$

Writing the first several terms of the sequence (2),

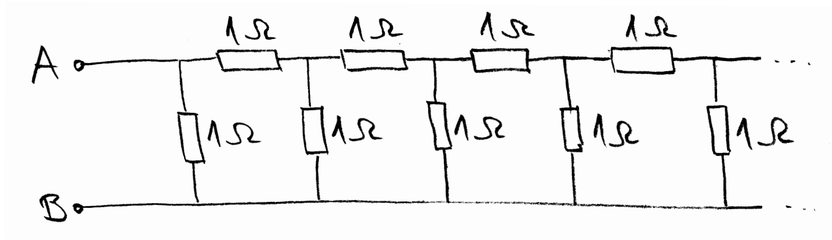
$$a_1 = 1, \quad a_2 = \tfrac{1}{1+1}, \quad a_3 = \frac{1}{1 + \frac{1}{1+1}}, \quad a_4 = \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1}}}, \quad a_5 = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1}}}}, \; \cdots \; ,$$

it is clear that that the sequence $\{a_n\}$ defined by (2) will converge in the limit $n \to \infty$ to the golden ratio σ defined by (1).

The first few terms of the sequence (2) are $a_1 = \frac{1}{1}$, $a_2 = \frac{1}{2}$, $a_3 = \frac{2}{3}$, $a_4 = \frac{3}{5}$, $a_5 = \frac{5}{8}$, $a_6 = \frac{8}{13}$, …, and one can recognize that $a_n = \dfrac{f_n}{f_{n+1}}$, where $\{f_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$ is the Fibonacci sequence defined on page 715 of the book.

One can show that the sequence $\{a_n\}$ defined by (2) converges, but – thankfully! – you don't have to to this. **The only thing you have to do in this problem is – taking the convergence of $\{a_n\}$ for granted – to find the exact value of the limit σ of the sequence $\{a_n\}$.** Example 14 on page 723 of the book can provide some inspiration. Having found σ, check that its numerical value is indeed the one given above.

Remark: Incidentally, in you connect infinitely many resistors, each of resistance $1\,\Omega$ (Ω is the symbol for Ohm), connected as shown in the figure below, then the resistance between points A and B will be exactly σ Ohms. Why? (This "Why?" is just food for thought, not a homework question.)



Food for Thought Problem.

Recall that in class we discussed the limit of the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n .$$

One can show (for example, by using the method of Mathematical Induction, see page A36 of the book) that the sequence $\{a_n\}$ is increasing and is bounded above by 3, so by the Monotone Sequence Theorem (page 722 of the book) it converges. Its limit,

$$e := \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.71828182845904523536028747135266249775724709369995957\ldots ,$$

is the base of the natural logarithms. Directly from the definition of e, prove the following limits:

(a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} = e^3$; in your derivation you will have to use the fact that the function $f(x) = x^3$ is continuous at $x = e$ – please specify where exactly you are using this;

(b) $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{n} = e^3$; hint: you can use that

$$\lim_{n\to\infty}\left(1+\tfrac{3}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\tfrac{1}{n/3}\right)^{n/3}\right]^3 = \lim_{m\to\infty}\left[\left(1+\tfrac{1}{m}\right)^m\right]^3 \overset{(!)}{=} \left[\lim_{m\to\infty}\left(1+\tfrac{1}{m}\right)^m\right]^3 ,$$

where in the step denoted by an exclamation mark we have used that the function $f(x) = x^3$ is continuous at $x = e$;

(c) $\lim_{n\to\infty}\left(\frac{n}{n+3}\right)^{n} = e^{-3}$; hint: using elementary algebra, you can rewrite the expression in such a form that you will be able to use the result of part (b); in some step you have to use the continuity of a certain function at certain point – please specify explicitly which function and at which point;

(d) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+53} = e$; hint: use that $\left(1+\frac{1}{n}\right)^{n+53} = \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{53}$, then take the limit and use some of the Limit Laws for Sequences on page 717.