MATH 3113 – Homework assigned on 10/30/13

Sec. 7.6: Problems 12, 15.

Hint to Problem 12: See Problem 1 from the homework assigned on 10/25/13.

Additional problem 1. Solve the following integrals:

$$\int_{-\infty}^{\infty} e^{-2t}\,\delta(t-3)\,dt\ , \quad \int_{-\infty}^{\infty} e^{-2t}\,\delta'(t-3)\,dt\ , \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-2t}\,\delta''(t-3)\,dt\ .$$

Additional problem 2.

(a) Find the transfer function $W(s)$ and the weight function $w(t)$ of the system described by the differential equation

$$x'' + 2x' + x = f(t)\ .$$

Assume that the initial conditions are $x(0) = 0$, $x'(0) = 0$.

(b) Use the convolution property to show that the solution of the initial value problem

$$x'' + 2x' + x = f(t)\ , \qquad x(0) = 0\ , \quad x'(0) = 0 \tag{1}$$

can be written as

$$x(t) = \int_0^t \tau\, e^{-\tau}\, f(t-\tau)\, d\tau\ .$$

(c) Apply the formula for $x(t)$ obtained in part (b) to find the solution of the initial value problem (1) in the case $f(t) = e^{-t}$.

(d) The system described by the initial value problem (1) can be interpreted physically. Namely, $x(t)$ can be thought of as the position of a particle with mass $m = 1$ (the term x'') attached to a spring of spring constant $k = 1$ (the term x), in the presence of damping (the corresponding term is $2x'$ – it is important to notice that its coefficient is positive!) and an external driving force $f(t)$. In the case considered in part (c), the external force $f(t) = e^{-t}$ decreases with t, so one can expect that after long enough time the particle will slow down.

The initial coordinate of the particle is $x(0) = 0$. Find the maximum value of the coordinate $x(t)$ of the particle,

$$x_{\max} = \max_{t\geq 0} x(t)\ .$$

At which moment t^* does the particle have coordinate $x_{\max}$?

Additional problem 3.

(a) Use the formula for translation on the s-axis (page 458 of the book) to find

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} .$$

(b) Using your result in (a) and the convolution property (pages 468–469), find

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s+2)^2}\right\} .$$

(c) Apply the formula for translation on the t-axis (page 475) to obtain

$$\mathcal{L}^{-1}\left\{e^{-s}\,\frac{1}{(s+2)^2}\right\} .$$

(d) Use your results in parts (a)–(c) to show that the solution of the initial value problem

$$x'' + 4x' + 4x = 1 + \delta(t-1) , \qquad x(0) = 0 , \quad x'(0) = 0$$

is

$$x(t) = \frac{1}{4}\left[1 - e^{-2t} - 2t\,e^{-2t}\right] + (t-1)\,e^{-2(t-1)}\,u(t-1) .$$

The graph of $x(t)$ is shown in the figure below. Note that the function $x(t)$ is continuous, but its slope has a jump at $t = 1$.