Additional problem assigned on 10/31/2017

Additional problem. Let f be a smooth (i.e., differentiable infinitely many times) function defined on the whole real line $\mathbb{R}$, which takes only strictly positive values, i.e., such that

$$f(x) > 0 \quad \text{for all} \quad x \in \mathbb{R} \ . \tag{1}$$

Define the function g as square root of f, i.e.,

$$g(x) := \sqrt{f(x)} \quad \text{for all} \quad x \in \mathbb{R} \ .$$

Clearly, the function g is well defined because of the condition (1) on f. Moreover, g is a composition of two smooth functions (namely, f and square root), so that it is smooth as well.

(a) Derive the formula

$$g'(x) = \frac{f'(x)}{2\sqrt{f(x)}} \ .$$

Which rules for differentiation did you need to derive this formula?

(b) Use your result in part (a) to prove that the functions f and g have the same critical numbers (i.e., that c is a critical number of f exactly when it is a critical number of g).

(c) Let c be a critical number of the function f. Prove that

$$g''(c) = \frac{f''(c)}{2\sqrt{f(c)}} \ . \tag{2}$$

To derive this formula, you first have to find $g''(x)$ for a general x, and then to set $x = c$ and to explain why your formula for $g''(x)$ simplifies to (2).

(d) Use (2) to show that g has a local maximum exactly at the same points where f has a local maximum, and that g has a local minimum exactly at the same points where f has a local minimum.

Remark. This problem justifies the trick used in Example 3 of Section 3.7, where we minimized the square of the distance d, instead of the distance itself (to avoid dealing with square roots).