MATH 3113 – Homework assigned on 11/8/13

Sec. 5.1: Problems 22, 27, 42.

Additional problem 1. Let $a > 0$ and $\epsilon > 0$ be positive numbers, and let the function $d_{a,\epsilon}(t)$ be defined as on page 494 of the book:

$$d_{a,\epsilon}(t) = \begin{cases} \frac{1}{\epsilon} & \text{if} \quad t \in [a, a+\epsilon] \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

(a) Directly from the definition of Laplace transform, find $D_{a,\epsilon}(s) := \mathcal{L}\{d_{a,\epsilon}(t)\}$, of the function $d_{a,\epsilon}(t)$.

(b) For a fixed value of s, find the limit $D_a(s) := \lim_{\epsilon \to 0} D_{a,\epsilon}(s)$.

(c) Compare your result in (b) with the Laplace transform $U_a(s)$ of the step function $u_a(t)$ for $a > 0$. What do you observe?

(d) What you observed in (c) was not an accident. Can you explain why?

Additional problem 2. Using the properties of the Laplace transform, one can show that the solution of the initial value problem

$$\begin{aligned} x'(t) + 3x(t) &= f(t) \\ x(0) &= 0 \end{aligned} \tag{1}$$

can be written in the form

$$x(t) = \int_0^t f(\tau)\, \mathrm{e}^{3(\tau - t)}\, \mathrm{d}\tau \ . \tag{2}$$

In this problem you will check that the function $x(t)$ defined in (2) indeed satisfies the initial value problem (1). You will need to use the following formula:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\phi(t)}^{\psi(t)} F(\tau, t)\, \mathrm{d}\tau = F(\psi(t), t)\, \psi'(t) - F(\phi(t), t)\, \phi'(t) + \int_{\phi(t)}^{\psi(t)} \frac{\partial F(\tau, t)}{\partial t}\, \mathrm{d}\tau \ . \tag{3}$$

(a) Show that $x(t)$ given by (2) satisfies the initial condition in (1).

(The problem continues on page 2)

(b) If you represent $x(t)$ from (2) in the form $\displaystyle\int_{\phi(t)}^{\psi(t)} F(\tau, t)\,\mathrm{d}\tau$, then write explicitly the functions $F(\tau, t)$, $\phi(t)$, and $\psi(t)$.

(c) Find explicitly $F(\psi(t), t)\,\psi'(t)$ and $F(\phi(t), t)\,\phi'(t)$.

(d) Find explicitly $\displaystyle\int_{\phi(t)}^{\psi(t)} \frac{\partial F(\tau, t)}{\partial t}\,\mathrm{d}\tau$.

(e) Using your results from (b), (c), and (d), prove that $x(t)$ given by (2) satisfies the differential equation in (1).