MATH 3113 – Homework assigned on 11/20/13

In this homework you will apply several different methods to solve the initial value problem

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x}\,, \qquad \text{where } \mathbf{A} = \begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix}, \tag{1}$$

$$\mathbf{x}(0) = \begin{pmatrix} -4 \\ 11 \end{pmatrix}. \tag{2}$$

Problem 1.

(a) Find the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$; let λ_1 be the smaller one: $\lambda_1 < \lambda_2$.

(b) Find the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\mathbf{A}$ corresponding to the eigenvalues λ_1 and λ_2 found in part (a).

(c) Write down the general solution of the ODE (1).

(d) Write down the solution of the IVP (1), (2).

Problem 2. In this problem you will find the solution of the IVP (1), (2) by using matrix exponentiation.

(a) Let the matrix $\mathbf{S}$ be a 2×2 matrix formed by writing the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the matrix $\mathbf{A}$ (found in Problem 1(b)) next to each other as column vectors:

$$\mathbf{S} = \Big(\big(\mathbf{v}_1\big) \; \big(\mathbf{v}_2\big) \Big).$$

(b) Write down the inverse of $\mathbf{S}$ by using the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(c) General theory says that the matrix

$$\mathbf{D} := \mathbf{S}^{-1}\,\mathbf{A}\,\mathbf{S}$$

will be diagonal. Find $\mathbf{D}$.

(d) What is $e^{\mathbf{D}t}$? (There is no need to do any computations, just write the answer.)

(e) Find $\mathrm{e}^{\mathbf{A}t}$ by using the fact that

$$\mathrm{e}^{\mathbf{A}t} = \mathrm{e}^{\mathbf{S}\,\mathbf{D}\,\mathbf{S}^{-1}\,t} = \mathbf{S}\,\mathrm{e}^{\mathbf{D}t}\,\mathbf{S}^{-1} .$$

(f) Use your result from part (e) to write the solution of the IVP (1), (2).

Problem 3.

(a) Rewrite the IVP (1), (2) as a system, without using vector notations.

(b) Solve the first equation of the system written in part (a), which contains only $x_1(t)$; the ODE for $x_1(t)$ is a very simple separable equation. Impose the initial condition $x_1(t)$ to find the function $x_1(t)$. In part (c) of this problem, use the expression for $x_1(t)$ obtained here.

(c) The second equation of the system written in part (a) now is a linear first-order ODE for $x_2(t)$. Solve this equation by using the methods from Lectures 4 and 5 (integrating factor, etc.). Impose the initial condition on $x_2(t)$ to find the solution of the IVP (1), (2).

Food For Thought Problem 4.
[You do not have to turn this problem in, but I strongly urge you to solve it!]
In this problem you will solve the IVP for $x_2(t)$ that you have already solved in Problem 3(c) by using different methods. Assume that the function $x_1(t)$ has already been found – use the expression for $x_1(t)$ that you have already obtained in Problems 1(c), 2(f), and 3(b).

(a) Solve the IVP for $x_2(t)$ by applying Laplace transform to both sides of the ODE and then computing the inverse Laplace transform directly (you may need to apply partial fractions; see Lectures 22 and 24).

(b) Solve the IVP for $x_2(t)$ by treating the ODE as a linear ODEs with constant coefficients. First you have to find the general solution of the associated homogeneous equation (by finding the roots of the characteristic equation), and then to apply the method of undetermined coefficients to find a particular solution of the non-homogeneous equation (see Lecture 17; you can download the handouts describing these methods from the class web-site).