MATH 2924 – Additional problem assigned on 11/20/15

Additional problem 1. Precise computation of the value of π has fascinated humankind for many centuries – see, e.g, the Wikipedia pages

`https://en.wikipedia.org/wiki/Chronology_of_computation_of_pi`

`https://en.wikipedia.org/wiki/Approximations_of_pi`

In the homework assignment from November 11 you derived François Viète's formula from 1593, and learned about one of the latest records in computing π by Shigeru Kondo used Alexander Yee (already broken on October 8, 2014, when 13.3 billion of digits of π were computed). In this homework you will derive another formula for efficient computation of π.

In 1671 the Scottish mathematician James Gregory (1638–1675) discovered the power series expansion of arctangent,

$$\arctan x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \ , \tag{1}$$

and in 1674 Gottfried Wilhelm Leibniz (1646–1716), one of the discoverers of differential and integral calculus, derived the formula

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) \tag{2}$$

by substituting $x = 1$ in (1). The infinite series (2) is called the *Gregory-Leibniz series*. The series (2) is mostly of theoretical interest because it converges very slowly.

In this problem you will derive another series for π that converges much faster; it was first obtained by the British astronomer John Machin (1680–1751), who used it in 1706 to compute 100 digits of π.

(a) Use the relations

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

to show that

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \ .$$

If $\alpha = \arctan x$ and $\beta = \arctan y$, use this identity to show that

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy} \ .$$

(b) It is easy to show that, if an invertible function f is odd, then f^{-1} is also odd: set $y = f(x)$ and use that $f(-x) = -f(x)$ for every x to obtain

$$f^{-1}(-y) = f^{-1}(-f(x)) = f^{-1}(f(-x)) = -x = -f^{-1}(y) .$$

Use this and the result from part (a) to show that

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy} .$$

(c) Use the relation from part (a) to fill in all details in the following chain of equalities:

$$\begin{aligned} 4\arctan\frac{1}{5} &= 2\left(\arctan\frac{1}{5} + \arctan\frac{1}{5}\right) = 2\arctan\frac{5}{12} \\ &= \arctan\frac{5}{12} + \arctan\frac{5}{12} = \arctan\frac{120}{119} . \end{aligned}$$

(d) Use the relation from part (b) to fill in all details in the following chain of equalities:

$$4\arctan\frac{1}{5} - \frac{\pi}{4} = \arctan\frac{120}{119} - \arctan 1 = \arctan\frac{1}{239}$$

which, together with the power series (1) yields the famous *Machin's formula*,

$$\begin{aligned} \pi &= 4\left(4\arctan\frac{1}{5} - \arctan\frac{1}{239}\right) \\ &= 4\left[4\left(\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \frac{1}{9\cdot 5^9} - \cdots\right)\right. \\ &\qquad \left. - \left(\frac{1}{239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 239^5} - \frac{1}{7\cdot 239^7} + \frac{1}{9\cdot 239^9} - \cdots\right)\right] . \end{aligned} \tag{3}$$

(e) Give an upper bound on the error of approximating the value of π by using the truncation

$$\pi = 4\left[4\left(\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7}\right) - \left(\frac{1}{239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 239^5} - \frac{1}{7\cdot 239^7}\right)\right] .$$

You can ignore the error that is coming from truncating the series expansion of $\arctan\frac{1}{239}$ because this error is negligible compared with the error coming from truncating the expansion of $\arctan\frac{1}{5}$.

(f) Look back at the Gregory-Leibniz series (2). To appreciate the speed of convergence of Machin's formula, estimate how many terms from the Gregory-Leibniz series we have to take into account to compute the value of π with accuracy 10^{-6}.