MATH 3113 – Homework assigned on 11/22/13

Problem 1. In this problem you will use several methods to solve the initial value problem

$$\begin{aligned} \mathbf{x}' &= \mathbf{A}\,\mathbf{x}\ , \qquad \text{where}\ \ \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \\ \mathbf{x}(0) &= \begin{pmatrix} -4 \\ 11 \end{pmatrix}\ . \end{aligned} \tag{1}$$

(a) Prove by directly computing the first several terms in the series expansion for $\mathrm{e}^{\mathbf{A}t}$ that

$$\mathrm{e}^{\mathbf{A}t} = \mathbf{I}\cosh t + \mathbf{A}\sinh t\ ,$$

and use this to find the solution of the IVP (1).

(b) Find the solution of the IVP (1) by computing the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.

(c) Show that from the linear system in (1) one can obtain that $x(t)$ satisfies the second-order ODE

$$x'' - x = 0\ ,$$

and use the method of Lecture 17 to solve it; don't forget to rewrite the initial conditions from (1) as initial conditions for $x(t)$. Once you have found the function $x(t)$, finding $y(t)$ is a matter of a simple differentiation.

Problem 2. In this problem you will find the solution of the initial value problem

$$\begin{aligned} x' &= y\ , \\ y' &= -x\ , \\ x(0) &= 3\ , \\ y(0) &= 5\ . \end{aligned} \tag{2}$$

by using a nice trick involving complex numbers

(a) Multiply the second ODE in (2) by i (where $\mathrm{i} := \sqrt{-1}$), and add the two ODEs.

(b) Let $z(t) := x(t) + \mathrm{i}y(t)$ be a new unknown function with values in $\mathbb{C}$. Show that the result of the manipulation in part (a) can be written as the following ODE for $z(t)$:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\mathrm{i}z\ .$$

What is the initial condition for $z(t)$?

(c) Solve the IVP for the new unknown function $z(t)$. Do not worry about the fact that you are working with complex numbers – the methods for real-valued functions that we have studied work without any change.

(d) Rewrite your result for $z(t)$ in terms of the original unknown functions $x(t)$ and $y(t)$.

Problem 3.

(a) Apply the method of partial fractions to solve the integral $\displaystyle\int \frac{\mathrm{d}y}{y^2 + y - 2}$.

(b) Use your result in part (a) to find the general solution of the ODE

$$t\,\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + y - 2\ .$$

(c) In the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 - \frac{2}{t^2}\ , \tag{3}$$

change the unknown function $x(t)$ to $y(t)$ by the transformation $x(t) = \dfrac{y(t)}{t}$. Use your result from part (b) to write the general solution of the ODE (3).

Problem 4. Apply the method of reduction of order to find the general solution of the ODE

$$y''(x) + \frac{1}{x^2}\,y'(x) - \frac{1}{x^3}\,y(x) = 0\ , \tag{4}$$

using that the function $y_1(x) = x$ is a solution of (4). The method of reduction of order was described in detail in the homework from 9/23/13 (assigned with Lecture 15), but here is the idea briefly: look for a solution of the ODE (4) of the form $y(x) = y_1(x)\,v(x)$ (i.e., in this particular problem, of the form $y(x) = x\,v(x)$) to obtain an ODE for the unknown function $v(x)$; after a "miraculous" cancellation, the ODE for v will only contain v'' and v', but not v, so that the substitution $w(x) := v'(x)$ will yield a first order ODE for $w(x)$ which typically is easier to solve. Then don't forget to go back from $w(x)$ to $v(x)$ and finally to the original unknown function $y(x)$.

Remark 1: In this problem the "old" notations are used, in which x is the independent variable, and y is the unknown function.

Remark 2: At some point you will have to solve the integral $\displaystyle\int \frac{1}{x^2}\,\mathrm{e}^{1/x}\,\mathrm{d}x$, which is a simple integral after an obvious substitution.

Remark 3: Since (4) is a second-order ODE, its general solution must have two independent constants.