MATH 2924 – Additional problems assigned on 12/07/15

Additional problem 1. The *Chebyshev polynomials (of the first kind)* are polynomials defined by

$$T_n(x) = \cos(n \arccos x) \ , \quad x \in [-1, 1] \ , \quad n = 0, 1, 2, 3, \dots \ ,$$

or, equivalently, by

$$T_n(\cos\theta) = \cos(n\theta) \ , \quad x \in [0, 2\pi) \ , \quad n = 0, 1, 2, 3, \dots \ .$$

(a) Find the explicit expressions for $T_0(x)$ and $T_1(x)$ directly from the definition.

(b) Use some of the relations

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

to show that

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta \ .$$

(c) Use the definition of the Chebyshev polynomials and relation established in (b) to prove the recurrence relation

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x) \ .$$

(d) Prove the *orthogonality relation* for Chebyshev polynomials

$$\int_{-1}^{1} T_k(x)\, T_n(x) \frac{\mathrm{d}x}{\sqrt{1 - x^2}} = \begin{cases} 0 & \text{if } n \neq m \ , \\ \pi & \text{if } n = m = 0 \ , \\ \frac{\pi}{2} & \text{if } n = m \neq 0 \ , \end{cases}$$

by using an appropriate (obvious) substitution. You may use (without deriving it) the result of Exersise 67 from Sec. 7.2 (on page 502 of Stewart's book).

Additional problem 2. The so-called *Gamma function* is defined as

$$\Gamma(x) := \int_0^\infty t^{x-1}\, \mathrm{e}^{-t}\, \mathrm{d}t \quad \text{for } x > 0 \ .$$

(a) Compute the value of $\Gamma(1)$ by a straightforward integration.

(b) Use integration by parts to prove that

$$\Gamma(x) = (x - 1)\, \Gamma(x - 1) \quad \text{for every } x > 1 \ .$$

(c) Use the results from parts (a) and (b) to prove by induction that

$$\Gamma(n) = (n - 1)! \ .$$