MATH 2934 **Additional problems, 04/12/2016** **Spring 2016**

Let f be a function of two variables, and ϕ and ψ be functions of a single variable. The integral

$$I(y) := \int_{\phi(y)}^{\psi(y)} f(x,y)\,\mathrm{d}x$$

is a function of a single variable, namely, y.

The following formula for the derivative of I holds:

$$I'(y) = f(\psi(y),y)\,\psi'(y) - f(\phi(y),y)\,\phi'(y) + \int_{\phi(y)}^{\psi(y)} \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x \tag{1}$$

Here is an example of using this formula to find the derivative of a function defined as a complicated integral:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y}\int_{\sqrt{y}}^{y^4} \frac{\mathrm{e}^{xy}}{x}\,\mathrm{d}x &= \left.\frac{\mathrm{e}^{xy}}{x}\right|_{x=y^4} \frac{\mathrm{d}}{\mathrm{d}y}\left(y^4\right) - \left.\frac{\mathrm{e}^{xy}}{x}\right|_{x=\sqrt{y}} \frac{\mathrm{d}}{\mathrm{d}y}\left(\sqrt{y}\right) + \int_{\sqrt{y}}^{y^4} \frac{\partial}{\partial y}\left(\frac{\mathrm{e}^{xy}}{x}\right)\mathrm{d}x \\
&= \frac{\mathrm{e}^{y^5}}{y^4}\,4y^3 - \frac{\mathrm{e}^{y^{3/2}}}{\sqrt{y}}\,\frac{1}{2\sqrt{y}} + \int_{\sqrt{y}}^{y^4} \mathrm{e}^{xy}\,\mathrm{d}x \\
&= \frac{4\mathrm{e}^{y^5}}{y} - \frac{\mathrm{e}^{y^{3/2}}}{2y} + \frac{1}{y}\,\mathrm{e}^{xy}\Big|_{x=\sqrt{y}}^{y^4} \\
&= \frac{4\mathrm{e}^{y^5}}{y} - \frac{\mathrm{e}^{y^{3/2}}}{2y} + \frac{\mathrm{e}^{y^5} - \mathrm{e}^{y^{3/2}}}{y} \\
&= \frac{5\mathrm{e}^{y^5}}{y} - \frac{3\mathrm{e}^{y^{3/2}}}{2y}
\end{aligned}$$

One can use this formula to solve some tricky definite integrals, as you will do in the problems below.

Additional problem 1. Find

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_{\sqrt{y}}^{y^4} \arctan\left(\mathrm{e}^x\right)\,\mathrm{d}x$$

Additional problem 2. Use equation (1) and the fact that, for any $y > 0$,

$$\int_0^\infty \frac{\mathrm{d}x}{x^2+y^2} = \frac{1}{y}\arctan\frac{x}{y}\Big|_{x=0}^{\infty} = \frac{\pi}{2y}$$

to prove that

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+y^2)^2} = \frac{\pi}{4y^3} \ .$$

Additional problem 3.

(a) Find the partial derivative

$$\frac{\partial}{\partial y}\left(\frac{x^y-1}{\ln x}\right) \ .$$

(b) If the function $F(y)$ is defined as

$$F(y) := \int_0^1 \frac{x^y-1}{\ln x}\,\mathrm{d}x \ ,$$

use equation (1) to show that

$$F'(y) = \frac{1}{y+1} \ .$$

(c) Use your result from part (b) to compute

$$\int_0^1 \frac{x^4-1}{\ln x}\,\mathrm{d}x \ .$$