MATH 3423 Homework 1 Due Fri, 8/30/13

Problem 1.

(a) Let f be an even function, g be an odd function, and h be their product defined by $h(x) = f(x)g(x)$. Is the function h even or odd? Prove your claim.

(b) Each function f can be represented as a sum of an even and an odd function:

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x) \ .$$

Write explicit expressions for $f_{\text{even}}(x)$ and $f_{\text{odd}}(x)$ in terms of $f(x)$ and $f(-x)$. If $f(x) = \mathrm{e}^x$, find $f_{\text{even}}(x)$ and $f_{\text{odd}}(x)$ (recall the definitions of the hyperbolic functions $\cosh x$ and $\sinh x$).

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathbb{R}$-valued function, and $\widehat{F}(\omega)$ be its Fourier transform. Show that $\widehat{F}(-\omega) = \widehat{F}(\omega)^*$ for any $\omega \in \mathbb{R}$ (where the star stands for complex conjugation). What can you say about $\widehat{F}(0)$?

Hint: Note that $f(t) \in \mathbb{R}$ for any $t \in \mathbb{R}$.

Problem 3. Use that the Fourier transform of the function $f(t) = \mathrm{e}^{-\alpha|t|}$ (where α is a positive real number) is equal to $\sqrt{\frac{2}{\pi}} \frac{\alpha}{k^2 + \alpha^2}$, as well as some property of Fourier transform (specify which property!) to find the Fourier transform of $t\mathrm{e}^{-\alpha|t|}$. (The result is given in the table on page 848.)

Problem 4. Use the fact that $\mathcal{F}\{\mathrm{e}^{-|x|}\} = \sqrt{\frac{2}{\pi}} \frac{1}{1 + k^2}$ and Parseval's theorem to show that

$$\int_0^\infty \frac{\mathrm{d}y}{(1 + y^2)^2} = \frac{\pi}{4} \ .$$

Problem 5.

(a) Directly from the definition of Fourier transform find $\mathcal{F}\{\delta(x)\}$.

(b) Explain why $x\delta(x) = 0$ for *any* $x \in \mathbb{R}$.

Hint: Let ϕ be any smooth function that is nonzero only on a bounded interval of $\mathbb{R}$. Show that

$$\int_{-\infty}^{\infty} x\delta(x)\, \phi(x)\, \mathrm{d}x = 0 \ .$$

(c) Let H be the *Heaviside function* (called also the *unit step function*),

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \ , \\ 1 & \text{if } x > 0 \ . \end{cases}$$

One can show that

$$\mathcal{F}\{H(x)\} = \sqrt{\frac{\pi}{2}}\, \delta(k) - \frac{i}{\sqrt{2\pi}\, k}$$

(you do not have to prove this).

Recall that $\delta = H'$, and use this and the expression for $\mathcal{F}\{H(x)\}$ to find $\mathcal{F}\{\delta(x)\}$.

Show that the result obtained here is the same as in (a).

Hint: You will need (b) to compare your results in (a) and (c).

(d) Using the expression for $\mathcal{F}\{\delta(x)\}$ and some property of Fourier transform, find $\mathcal{F}\{\delta'(x)\}$ and $\mathcal{F}\{\delta''(x)\}$.

Problem 6. Consider the following initial value problem (where v is a positive real constant):

$$\frac{1}{v}\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \ , \qquad x \in \mathbb{R} \ , \quad t \geq 0 \ ,$$
$$u(x,0) = f(x) \ .$$

(a) Let the Fourier transform with respect to x of $u(x,t)$ be

$$\hat{U}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, \mathrm{e}^{-ikx}\, \mathrm{d}x$$

and its inverse be

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{U}(k,t)\, \mathrm{e}^{ikx}\, \mathrm{d}k \ .$$

Find $\hat{U}(k,t)$ for the initial value problem.

(b) Use some of the properties of the Fourier transform (state explicitly which property you have used!) to show that the solution of the initial value problem is

$$u(x,t) = f(x+vt) \ .$$