MATH 3423 **Homework 1** **Due Fri, 09/02/16**

Problem 1. Give direct proofs of the following identities:

(a) $(z_1 z_2)^* = z_1^* \, z_2^*$;

(b) $\left(\dfrac{1}{z}\right)^* = \dfrac{1}{z^*}$;

(c) $|z_1 z_2| = |z_1| \, |z_2|$.

Problem 2. In the complex plane, describe in words and sketch the domain D given by the inequalities

$$1 < |z + \mathrm{i}| \leq 3 \ .$$

Denote the boundaries that do not belong to D with dashed lines, and the boundaries that belong to D with solid lines.

Problem 3. Evaluate $\operatorname{Re} \dfrac{5 + \mathrm{i}}{2 - \mathrm{i}}$.

Problem 4. Express the following function $w(z) = \dfrac{z^*}{z}$ in the form $w(z) = u(x, y) + \mathrm{i}\, v(x, y)$ (where u and v are real-valued functions of two real variables).

Problem 5. Determine the modulus and the principal argument of the complex numbers

(a) $2 - 2\mathrm{i}$;

(b) $-\mathrm{i}$;

(c) $-3 + \sqrt{3}\,\mathrm{i}$.

Hint: Drawing pictures in a problem like this is very helpful.

Problem 6. Express the following complex numbers in Cartesian coordinates (i.e., in the form $z = x + \mathrm{i}y$):

(a) $\mathrm{e}^{\ln 2 - (\pi/4)\mathrm{i}}$;

(b) $6\, \mathrm{e}^{2\pi \mathrm{i}/3}$.

Problem 7. Evaluate $\left(\sqrt{3} + \mathrm{i}\right)^{14}$; write your result in the form $\left(\sqrt{3} + \mathrm{i}\right)^{14} = x + \mathrm{i}y$.

Hint: First write $\sqrt{3} + \mathrm{i}$ in polar coordinates.

Problem 8. Find all 4th roots of i.

Problem 9. Starting with $\int_0^\infty \mathrm{e}^{-\beta^2 y^2}\,\mathrm{d}y = \frac{\sqrt{\pi}}{2\beta}$, let $\beta = \frac{1-\mathrm{i}}{\sqrt{2}}$ (and notice that $\beta^2 = -\mathrm{i}$) to show that

$$\int_0^\infty \cos\bigl(x^2\bigr)\,\mathrm{d}x = \int_0^\infty \sin\bigl(x^2\bigr)\,\mathrm{d}x = \sqrt{\frac{\pi}{8}}\ .$$

Problem 10. Directly from the definition of $\sinh z$, show that the Taylor expansion of $\sinh z$ around 0 is

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\ .$$