MATH 4073 **Homework 1** **Due Thu, 09/03/09**

Problem 1. Consider the function

$$f(x) = \mathrm{e}^{2x/\pi} + (1 - \mathrm{e}) \sin x \ .$$

(a) Use Rolle's Theorem to show that the derivative of f vanishes (i.e., becomes equal to zero) at least once in the interval $[0, \frac{\pi}{2}]$, without computing f' explicitly.

(b) In the rest of this problem you will give another solution of what you already proved in part (a), and, in addition, will show that the point where f' vanishes is unique. Start by finding the derivative of f explicitly.

(c) Use some of the theorems from Section 1.1 to prove that the equation $f'(x) = 0$ has at least one solution in the interval $[0, \frac{\pi}{2}]$.

Hint: Find the values of $f'(0)$ and $f'(\frac{\pi}{2})$.

(d) Show that the solution of $f'(x) = 0$ whose existence was proved in part (c) is in fact unique.

Hint: Take the derivative of the left-hand side of the equation $f'(x) = 0$ and stare at it long enough.

Problem 2. Derive the Mean Value Theorem from Rolle's Theorem.

Hint: Define the function $g(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}\,(x - a)$, and apply Rolle's Theorem to g.

Problem 3. Find the limit

$$\lim_{x \to 0} \frac{\exp(x^2) - \cos x}{x^2}$$

by using the Taylor expansions of the functions $\mathrm{e}^{x^2} = 1 + \frac{1}{1!}(x^2) + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \cdots$ and $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$.

Remark: Applying the L'Hospital rule will be more difficult and error-prone.

Problem 4. In this problem you will use Taylor's formula to approximate the value of $\arctan 1.01$.

(a) Write the second-degree Taylor polynomial, $P_2(x)$, for the function $f(x) = \arctan x$ around $x_0 = 1$. You may use that $f(1) = \frac{\pi}{4}$, $f'(x) = \frac{1}{1+x^2}$ and $f''(x) = -\frac{2x}{(1+x^2)^2}$.

(b) Find the numerical value of $P_2(1.01)$.

(c) Write the remainder term $R_2(x)$ and find the maximum possible value of $|R_2(1.01)|$. You may use that $f'''(x) = \frac{6x^2-2}{(1+x^2)^3}$.

Hint: This is a bit more complicated. According to the formula for the remainder term, the error, $|R_2(1.01)|$, in approximating $f(1.01)$ by $P_n(1.01)$ cannot exceed

$$\max_{z \in [1, 1.01]} \left| \frac{1}{(n+1)!} f^{(n+1)}(z)(1 - 1.01)^{n+1} \right| ,$$

where z is a point between $x_0 = 1$ and $x = 1.01$. In the book the point where $f^{(n+1)}$ is evaluated was denoted by $\xi(x)$, but the problem is that we do not know $\xi(x)$, so we take the maximum possible value of $|f^{(n+1)}(z)|$ over the whole interval between x_0 and x, which will certainly be no less than $|f^{(n+1)}(\xi(x))|$, whatever $\xi(x)$ is. To find $\max_{z\in[1,1.01]} |f^{(n+1)}|$, you have to find the value of z where $f^{(n+1)}$ reaches its maximum value, which can be found by looking at the $f^{(n+2)}$. In this particular problem, you have to find $\max_{z\in[1,1.01]} |f^{(3)}(z)|$, and it will be helpful to look at $f^{(4)}(x) = -\frac{24x(x^2-1)}{(1+x^2)^4}$. In fact, it is enough to note that $f^{(4)}$ is non-positive on the interval $[1, 1.01]$, which implies that $f^{(3)}$ reaches its maximum at one of the endpoints of this interval – which one?

(d) Compute the true numerical value of the absolute error, $|P_2(1.01) - f(1.01)|$. Compare the true value of $|P_2(1.01) - f(1.01)|$ with the exact upper bound for the error obtained in part (c). Discuss briefly.

Problem 5. In class we discussed some methods for computing the numerical value of the integral

$$I = \int_0^{1/2} \frac{1}{1+x^4}\,\mathrm{d}x ,$$

whose exact numerical value is

$$I = \frac{1}{4\sqrt{2}}\left[2\arctan(1+\tfrac{1}{\sqrt{2}}) - 2\arctan(1-\tfrac{1}{\sqrt{2}}) + \ln\tfrac{33+20\sqrt{2}}{17}\right] = 0.49395805107743802332\ldots .$$

In class we found approximate values of I by integrating the Taylor expansion of $f(x) = \frac{1}{1+x^4}$ around $x_0 = 0$ (and using a nice trick with the formula for the sum of a geometric series). We also noticed that the function f is decreasing on $[0, \frac{1}{2}]$, so that

$$\tfrac{16}{17} = f(\tfrac{1}{2}) \le f(x) \le f(0) = 1 \quad \text{for all } x \in [0, \tfrac{1}{2}] ,$$

integrating which implied the exact bounds

$$\tfrac{8}{17} = 0.47058823529411764706\ldots \le I \le \tfrac{1}{2} = 0.5 .$$

In this problem you will extend this problem to get a tighter lower bound on I.

(a) Prove that the function $f(x) = \frac{1}{1+x^4}$ is concave on the interval $[0, \frac{1}{2}]$.

(b) Draw a sketch and write a short explanation to convince me that the area of the trapezoid with vertices $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, f(\frac{1}{2}))$, $(0, f(0))$ is strictly smaller than I, and strictly larger than the value $\frac{8}{17}$ found in class.

(c) Find the exact value of the area of the trapezoid from part (b) – as you explained in (b), it will be a better lower bound than $\frac{8}{17}$.

(d) Now draw a sketch to convince me that the area of the pentagon with vertices $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, f(\frac{1}{2}))$, $(\frac{1}{4}, f(\frac{1}{4}))$, $(0, f(0))$ is a rigorous lower bound on I that will be better (i.e., larger) than the value found in part (c).

(e) Find the lower bound described in part (d), and compare it with the previous lower bounds.

Hint: This is very easy because the pentagon can be divided into two trapezoids.

(f) Finally, compute the absolute and relative error if we approximate the exact value of I with the lower bound found in part (e).

Problem 6.

(a) Convert the number 11010110011.001_2 to base 10.

(b) Convert the number 417_{10} to base 2.

(c) Convert the number $\mathrm{AF2}_{16}$ to base 2.

(d) Convert the number 11010110011.001_2 to base 16.

Hint: Since $16 = 2^4$, it is easy to convert from base 2 to base 16: divide the binary number in groups of 4 digits to the left and to the right of the decimal point: $0110|1011|0011.|0010_2$ (padding with zeros if needed), and then convert each group of four binary digits (representing an integer from 0_{10} to 15_{10}, i.e., from 0_{16} to F_{16}) to a hexadecimal form.