MATH 4073 **Homework 1** **Due Fri, 09/03/10**

Problem 1. Prove that the equation $(x-2)^2 = \ln x$ has at least one solution in the interval $[1,2]$.

Hint: Write the equation in the form $f(x) = 0$ for an appropriately defined function $f(x)$, and apply some of the fundamental theorems of Calculus.

Problem 2. Prove that the first derivative of the function

$$f(x) = x\,(x^3 - 5x^2 + 6x + 4)\cos(2\pi x)$$

vanishes (i.e., becomes equal to 0) at least once in the interval $[-1,2]$.

Hint: You can do this without computing the derivative!

Problem 3. Applying the L'Hospital rule to find limits of ratios where both the numerator and the denominator tend to zero is sometimes long and error-prone. Use the Taylor expansions

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\ , \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\ ,$$

to compute the value of the limit

$$\lim_{x\to 0} \frac{\cos x + \frac{1}{2}\ln(1+x^2) - 1}{x^4}\ .$$

Problem 4. In this problem you will use Taylor's Theorem to approximate the value of $\sqrt{17}$.

(a) Write the second-degree Taylor polynomial,

$$P_2(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2$$

for the function $f(x) = \sqrt{x}$ around $x_0 = 16$. You have to find explicitly the numerical values of the coefficients of $P_2(x)$; there is no need to expand the factors $(x-x_0)^j$.

Hint: The answer is: $P_2(x) = 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2$, but I want to see your derivations.

(b) Find the numerical value of $P_2(17)$.

(c) Show that the remainder term is $R_2(x) = \frac{f'''(\xi(x))}{3!}(x-x_0)^3 = \frac{1}{16\,[\xi(x)]^{5/2}}\big(x-16\big)^3$, and find the maximum possible value of $|R_2(17)|$. Here $\xi(x)$ is a number between $x_0 = 16$ and $x = 17$; this number is unknown, so to find the maximum possible value of $|R_2(17)|$, you have to allow $\xi(x)$ to be *anywhere* between 16 and 17. The maximum possible value of $|R_2(17)|$ is a *rigorous* upper bound on the size of the error if you replace the exact value $f(17) = \sqrt{17}$ with its approximation, $P_2(17)$.

(d) Compute the true numerical value of the so-called *absolute error*, $|P_2(17) - \sqrt{17}|$ (the absolute error is the absolute value of the difference between the exact and the approximate values). Compare the true value of $|P_2(17) - \sqrt{17}|$ with the upper bound for the error obtained in part (c). Discuss briefly your observations.

Problem 5. In this problem you will develop some methods for estimating and computing the numerical value of the integral $I = \displaystyle\int_0^{1/2} \frac{1}{1+x^7}\,\mathrm{d}x$. If $s_j = \sin\frac{j\pi}{7}$ and $c_j = \cos\frac{j\pi}{7}$, then the exact numerical value of the integral can be shown to be

$$\begin{aligned} I &= \frac{\pi}{49}\left(5s_1 + s_3 - 3s_5\right) + \frac{1}{7}\big[(-1 + 2c_3 + 2c_5)\ln 2 + \ln 3 + c_1 \ln 4\big] \\ &\quad -\frac{1}{7}\sum_{j=1,3,5}\Big[c_j \ln\left(5 - 4c_j\right) - 2s_j \arctan\tfrac{2c_j - 1}{2s_j}\Big] = 0.49951374248182774179996 71\ldots\ . \end{aligned}$$

In all parts of the problem below we use the notation $f(x) = \frac{1}{1+x^7}$. You may find useful that $f'(x) = -\frac{7x^6}{(1+x^7)^2}$ and $f''(x) = \frac{14x^5(4x^7-3)}{(1+x^7)^3}$.

(a) Show that f is decreasing on $[0, \frac{1}{2}]$ and use this to prove the rigorous bounds $\frac{64}{129} \le I \le \frac{1}{2}$.

(b) Prove that the function f is concave down on the interval $[0, \frac{1}{2}]$.

(c) Draw a sketch and write a short explanation to convince me that the area of the trapezoid with vertices $(0,0)$, $(\frac{1}{2},0)$, $(\frac{1}{2}, f(\frac{1}{2}))$, $(0, f(0))$ is strictly smaller than I, and strictly larger than $\frac{64}{129}$.

(d) Find the exact value of the area of the trapezoid from part (c). As you showed in part (c), it will be a better lower bound than $\frac{64}{129}$.

(e) Now draw a sketch to convince me that the area of the pentagon with vertices $(0,0)$, $(\frac{1}{2},0)$, $(\frac{1}{2}, f(\frac{1}{2}))$, $(\frac{1}{4}, f(\frac{1}{4}))$, $(0, f(0))$ is a rigorous lower bound on I that will be better (i.e., larger) than the value found in part (d).

(f) Find the lower bound described in part (e), and compare it with the previous lower bounds.

Hint: This is very easy because the pentagon can be divided into two trapezoids.

(g) Show that the Taylor series of f is $f(x) = 1 - x^7 + x^{14} - x^{21} + \cdots$.

(h) One can integrate the Taylor series from part (g) term by term to obtain that $\int_0^{1/2}(1 - x^7 + x^{14} - x^{21} + \cdots)\,\mathrm{d}x = \frac{1}{2} - \frac{1}{8\cdot 2^8} + \frac{1}{15\cdot 2^{15}} - \frac{1}{22\cdot 2^{22}} + \cdots$, which can be used to obtain better and better approximations to the true value of I. Assume that we only use the first two terms to obtain the approximate value $\int_0^{1/2}(1 - x^7)\,\mathrm{d}x = \frac{1}{2} - \frac{1}{8\cdot 2^8} = \frac{1023}{2048} = 0.49951171875$. Use the theorem about the truncation error for alternating series to give a rigorous upper bound on the error in approximating the true value of I by $\frac{1023}{2048}$. Then compute the true error, $|I - \frac{1023}{2048}|$, and make sure that it is smaller than the rigorous bound you just obtained.

Problem 6.

(a) Convert the number 1101011001.001_2 to base 10.

(b) Convert the number 287_{10} to base 2.

(c) Convert the number 11010110011.001_2 to base 16.

(d) Convert the number $\mathrm{B2F}_{16}$ to base 2.