MATH 4073 Homework 1 Due Tue, 8/27/13

Problem 1. Consider the function

$$f(x) = \mathrm{e}^{2x/\pi} + (1 - \mathrm{e}) \sin x \ .$$

(a) Use Rolle's Theorem to show that the derivative of f vanishes (i.e., becomes equal to zero) at least once in the interval $[0, \frac{\pi}{2}]$, without computing f' explicitly.

(b) In the rest of this problem you will give another solution of what you already proved in part (a), and, in addition, will show that the point where f' vanishes is unique. Start by finding the derivative of f explicitly.

(c) Use some of the theorems mentioned in class to prove that the equation $f'(x) = 0$ has at least one solution in the interval $[0, \frac{\pi}{2}]$.

Hint: Find the values of $f'(0)$ and $f'(\frac{\pi}{2})$.

(d) Show that the solution of $f'(x) = 0$ whose existence was proved in part (c) is in fact unique.

Hint: Take the derivative of the left-hand side of the equation $f'(x) = 0$ and stare at it long enough.

Problem 2. Find the limit

$$\lim_{x \to 0} \frac{\exp(x^2) - \cos x}{x^2}$$

by using the Taylor expansions of the functions $\mathrm{e}^{x^2} = 1 + \frac{1}{1!}(x^2) + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \cdots$ and $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$.

Remark: Applying the L'Hospital rule will be more difficult and error-prone.

Problem 3. In this problem you will use Taylor's formula to approximate the value of $\arctan 1.01$.

(a) Write the second-degree Taylor polynomial, $P_2(x)$, for the function $f(x) = \arctan x$ around $x_0 = 1$. You may use that $f(1) = \frac{\pi}{4}$, $f'(x) = \frac{1}{1+x^2}$ and $f''(x) = -\frac{2x}{(1+x^2)^2}$.

(b) Show that the numerical value of $P_2(1.01)$ is $0.790373163397\ldots$.

(c) Write the remainder term $R_2(x)$ and find the maximum possible value of $|R_2(1.01)|$. You may use that $f'''(x) = \frac{6x^2-2}{(1+x^2)^3}$.

Hint: This is a bit more complicated. According to the formula for the remainder term, the error, $|R_2(1.01)|$, in approximating $f(1.01)$ by $P_n(1.01)$ cannot exceed

$$\max_{z \in [1, 1.01]} \left| \frac{1}{(n+1)!} f^{(n+1)}(z) (1 - 1.01)^{n+1} \right| \ ,$$

where z is a point between $x_0 = 1$ and $x = 1.01$. Since we do not know the point z where $f^{(n+1)}$ is evaluated, we take the maximum possible value of $|f^{(n+1)}(z)|$ over the whole interval between x_0 and x, which will certainly be no less than $|f^{(n+1)}(z_1)|$ for any particular point $z \in [1, 1.01]$. To find $\max\limits_{z\in[1,1.01]} |f^{(n+1)}(z)|$, you have to find the value of z where $f^{(n+1)}$ reaches its maximum value, which can be found by looking at the $f^{(n+2)}$. In this particular problem, you have to find $\max\limits_{z\in[1,1.01]} |f^{(3)}(z)|$, and it will be helpful to look at $f^{(4)}(x) = -\frac{24x(x^2-1)}{(1+x^2)^4}$. In fact, it is enough to note that $f^{(4)}$ is non-positive on the interval $[1, 1.01]$, which implies that $f^{(3)}$ reaches its maximum at one of the endpoints of this interval – which one?

(d) Compute the true numerical value of the absolute error, $|P_2(1.01) - f(1.01)|$. Compare the true value of $|P_2(1.01) - f(1.01)|$ with the exact upper bound for the error obtained in part (c). Discuss briefly.

Problem 4. In this problem you will develop some methods for estimating and computing the numerical value of the integral $I = \displaystyle\int_0^{1/2} \frac{1}{1+x^7}\,\mathrm{d}x$. If $s_j = \sin\frac{j\pi}{7}$ and $c_j = \cos\frac{j\pi}{7}$, then the exact numerical value of the integral can be shown to be

$$
\begin{aligned}
I \;=\; & \frac{\pi}{49}\,(5s_1 + s_3 - 3s_5) + \frac{1}{7}\big[(-1 + 2c_3 + 2c_5)\ln 2 + \ln 3 + c_1 \ln 4\big] \\
& -\frac{1}{7}\sum_{j=1,3,5}\Big[c_j \ln(5 - 4c_j) - 2s_j \arctan\tfrac{2c_j-1}{2s_j}\Big] = 0.49951374248182774179996 71\ldots\ .
\end{aligned}
$$

In all parts of the problem below we use the notation $f(x) = \frac{1}{1+x^7}$. You may find useful that $f'(x) = -\frac{7x^6}{(1+x^7)^2}$ and $f''(x) = \frac{14x^5(4x^7-3)}{(1+x^7)^3}$.

(a) Show that f is decreasing on $[0, \frac{1}{2}]$, which implies that

$$1 = f(0) \ge f(x) \ge f(\tfrac{1}{2}) = \tfrac{128}{129}\ .$$

Integrate these inequalities and use the monotonicity properties of integration – i.e., the fact that if $\phi(x) \le \psi(x)$ for any $x \in [a, b]$, then $\displaystyle\int_a^b \phi(x)\,\mathrm{d}x \le \int_a^b \psi(x)\,\mathrm{d}x$ – to prove the rigorous bounds $\frac{64}{129} \le I \le \frac{1}{2}$.

(b) Prove that the function f is concave down on the interval $[0, \frac{1}{2}]$.

(c) Draw a sketch and write a short explanation to convince me that the area of the trapezoid with vertices $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, f(\frac{1}{2}))$, $(0, f(0))$ is strictly smaller than I, and strictly larger than $\frac{64}{129}$.

(d) Find the exact value of the area of the trapezoid from part (c). As you showed in part (c), it will be a better lower bound than $\frac{64}{129}$.

(e) Now draw a sketch to convince me that the area of the pentagon with vertices $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, f(\frac{1}{2}))$, $(\frac{1}{4}, f(\frac{1}{4}))$, $(0, f(0))$ is a rigorous lower bound on I that will be better (i.e., larger) than the value found in part (d).

(f) Find the lower bound described in part (e), and compare it with the previous lower bounds.

Hint: This is very easy because the pentagon can be divided into two trapezoids.

(g) Show that the Taylor series of f is $f(x) = 1 - x^7 + x^{14} - x^{21} + \cdots$.
Hint: This can be done very easily by noticing that $\frac{1}{1+x^7} = \frac{1}{1-(-x^7)}$ and applying the formula for the sum of a geometric series, $1 + q + q^2 + q^3 + \cdots = \frac{1}{1-q}$, valid for $|q| < 1$.

(h) One can integrate the Taylor series from part (g) term by term to obtain that

$$\int_0^{1/2} (1 - x^7 + x^{14} - x^{21} + \cdots)\,\mathrm{d}x = \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \cdots ,$$

which can be used to obtain better and better approximations to the true value of I. Assume that we only use the first two terms to obtain the approximate value

$$\int_0^{1/2} (1 - x^7)\,\mathrm{d}x = \frac{1}{2} - \frac{1}{8 \cdot 2^8} = \frac{1023}{2048} = 0.49951171875 .$$

To give a theoretical bound on the accuracy of this approximation, one can use the following theorem about series the signs of whose terms alternate:

Theorem. Consider the series

$$\sum_{k=1}^{\infty} (-1)^k b_k = b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + \cdots ,$$

and let $S_n := \sum_{k=1}^{n} (-1)^k b_k$ be the nth partial sum. Let the series converge to the number S, and assume that the numbers b_k satisfy the following properties:

$$b_k \geq 0 \;\text{ and }\; b_k \geq b_{k+1} \text{ for any } k , \quad \lim_{k\to\infty} b_k = 0 ,$$

then the truncation error, $|S - S_n|$, satisfies the bound $|S - S_n| \leq b_{n+1}$.

Use this theorem to give a rigorous upper bound on the error in approximating the true value of I by $\frac{1023}{2048}$. Then compute the true error, $|I - \frac{1023}{2048}|$, compare it with the rigorous upper bound.

Problem 5.

(a) Convert the number 11010110011.001_2 to base 10.

(b) Convert the number 417_{10} to base 2.

(c) Convert the number $\mathrm{AF2}_{16}$ to base 2.

(d) Convert the number 11010110011.001_2 to base 16.

Hint: Since $16 = 2^4$, it is easy to convert from base 2 to base 16: divide the binary number in groups of 4 digits to the left and to the right of the decimal point: $0110|1011|0011.|0010_2$ (padding with zeros if needed), and then convert each group of four binary digits (representing an integer from 0_{10} to 15_{10}, i.e., from 0_{16} to F_{16}) to a hexadecimal form.