

Problem 1. Two norms, $\|\cdot\|$ and $\|\cdot\|'$, on the same vector space V are said to be *equivalent* if there exist positive constants A and B such that

$$A\|\mathbf{u}\| \leq \|\mathbf{u}\|' \leq B\|\mathbf{u}\| \quad \text{for any } \mathbf{u} \in V .$$

Consider the vector space \mathbb{R}^n with the following norms defined on it:

$$\|\mathbf{u}\|_1 := \sum_{j=1}^n |u_j| , \quad \|\mathbf{u}\|_2 := \left(\sum_{j=1}^n |u_j|^2 \right)^{1/2} , \quad \|\mathbf{u}\|_\infty := \max_{1 \leq j \leq n} |u_j| .$$

One can prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on \mathbb{R}^n are equivalent as follows: for an arbitrary vector $\mathbf{u} \in \mathbb{R}^n$ we have

$$\|\mathbf{u}\|_1 = \sum_{j=1}^n |u_j| \leq \sum_{j=1}^n \max_{1 \leq k \leq n} |u_k| \leq n \max_{1 \leq k \leq n} |u_k| = n\|\mathbf{u}\|_\infty ,$$

(where we used the obvious fact that $|u_j| \leq \max_{1 \leq k \leq n} |u_k|$ for any $j = 1, \dots, n$), and

$$\|\mathbf{u}\|_\infty = \max_{1 \leq k \leq n} |u_k| \leq \sum_{j=1}^n |u_j| = \|\mathbf{u}\|_1 ,$$

hence

$$\|\mathbf{u}\|_\infty \leq \|\mathbf{u}\|_1 \leq n\|\mathbf{u}\|_\infty ,$$

which proves our claim (for the choice of constants $A = 1$, $B = n$).

- (a) Prove that the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent.
 (b) Prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Hint: From the fact that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent, and the fact that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent (proved in part (a)), you can solve part (b) without any additional calculations.

Problem 2. Many theorems that hold in finite-dimensional spaces are not true in infinite-dimensional spaces. One can think of the real infinite-dimensional space \mathbb{R}^∞ as the space of infinite sequences: $\mathbf{u} = (u_1, u_2, u_3, \dots)$, where u_j are real numbers ($j \in \mathbb{N} := \{1, 2, 3, \dots\}$). In this space we can define the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ as usual:

$$\|\mathbf{u}\|_1 := \sum_{j \in \mathbb{N}} |u_j| , \quad \|\mathbf{u}\|_2 := \left(\sum_{j \in \mathbb{N}} |u_j|^2 \right)^{1/2} , \quad \|\mathbf{u}\|_\infty := \sup_{j \in \mathbb{N}} |u_j| .$$

Here $\sup_{j \in \mathbb{N}} a_j$ (the “supremum”) is the smallest number a such that $a_j \leq a$ for all $j \in \mathbb{N}$. The supremum over a finite set of real numbers is the same as the maximum over this set. For an infinite set, however, the set may not have a maximum, but it always has a supremum (which may be finite or infinite); for example, the set $\{5 - 1, 5 - \frac{1}{2}, 5 - \frac{1}{3}, 5 - \frac{1}{4}, \dots, 5 - \frac{1}{k}, \dots\}$ has a supremum (equal to 5), but does not have a maximum (because none of the elements of the set is *equal* to 5).

(a) Give an explicit example of a sequence \mathbf{u} such that $\|\mathbf{u}\|_\infty < \infty$, but $\|\mathbf{u}\|_1$ is infinite.

Hint: How about $\mathbf{u} = (1, 1, 1, \dots)$?

(b) Give an explicit example of a sequence \mathbf{u} such that $\|\mathbf{u}\|_\infty < \infty$, but $\|\mathbf{u}\|_2$ is infinite.

(c) Give an explicit example of a sequence \mathbf{u} such that $\|\mathbf{u}\|_2 < \infty$, but $\|\mathbf{u}\|_1$ is infinite.

Hint: Think how you can use the following facts:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}, \quad \sum_{j=1}^{\infty} \frac{1}{j} = \infty.$$

Problem 3. In this problem, a “polynomial” means a polynomial of a real variable with real coefficients (so that both x and $P(x)$ are real numbers). As discussed in class, the polynomials of order no higher than n form a linear space with respect to the addition of polynomials and multiplication of a polynomial by a real number as follows: if P and Q are polynomials of degree $\leq n$ and $\alpha \in \mathbb{R}$, then the polynomials $P + Q$ and αP are defined as

$$(P + Q)(x) := P(x) + Q(x), \quad (\alpha P)(x) := \alpha P(x).$$

Let $V_n(a, b; w(x))$ stand for the linear space of polynomials defined on the interval with left end a and right end b (at each end, the interval can be open or closed; a and b can be finite or infinite) of degree no greater than n endowed with the inner product

$$\langle P, Q \rangle = \int_a^b P(x) Q(x) w(x) dx.$$

Samer defined a family of polynomials which he denoted (very modestly!) by S_0, S_1, S_2, \dots . These polynomials satisfy the following conditions:

- (i) the polynomial S_k is of degree k ;
- (ii) the coefficient of x^k in S_k is equal to 1 (such polynomials are called *monic*);
- (iii) the polynomials $S_0, S_1, S_2, \dots, S_n$ form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = e^{-x})$.

In the solution of this problem the following identity will be handy:

$$\int_0^{\infty} x^k e^{-x} dx = k!$$

(where, by definition, $0! = 1$).

- Clearly, $S_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial S_1 of degree 1 that is orthogonal to S_0 (i.e., such that $\langle S_1, S_0 \rangle = 0$).
- Find the only monic quadratic polynomial S_2 that is orthogonal to both S_0 and S_1 .
- Show that the polynomial $P(x) = x^2 + 3$ can be represented as a linear combination of the polynomials S_0 , S_1 and S_2 as follows: $P = S_2 + 4S_1 + 5S_0$.
- Show by direct integration that $\langle S_0, S_0 \rangle = 1$, $\langle S_1, S_1 \rangle = 1$, $\langle S_2, S_2 \rangle = 4$.
- Find the orthogonal projection, $\text{proj}_{S_0+2S_1} P$, of the polynomial $P(x) = x^2 + 3$ onto the “straight line”

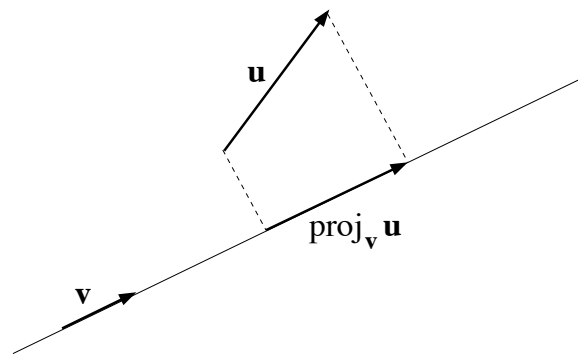
$$\ell := \{t(S_0 + 2S_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; e^{-x})$. If you have solved part (c), then finding this orthogonal projection should be easy.

Hint: If \mathbf{u} and \mathbf{v} are vectors in the inner product linear space V , then the orthogonal projection of the vector \mathbf{u} onto the straight line in the direction of \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$$

– see the picture below.



- Finally, let $\tilde{S}_k := \mu_k S_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\|\tilde{S}_k\| := \sqrt{\langle \tilde{S}_k, \tilde{S}_k \rangle},$$

of the polynomial \tilde{S}_k is 1. Find the explicit expressions for $\tilde{S}_0(x)$, $\tilde{S}_1(x)$, and $\tilde{S}_2(x)$.