MATH 4443 **Homework 1** **Due 1/25/18 (Thursday)**

Problem 1. Directly from the definition of the limit of a sequence, find $\lim_{n\to\infty} \frac{3n^5}{3n^5 - \sin n}$.

Problem 2. Use the definition of the limit of a sequence, to prove if $\lim_{n\to\infty} a_n = a > 0$, then there exists a number N such that $a_n > 0$ for all $n > N$.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Directly from the definition of continuity, prove that if $f(b) = a > 0$, then there exists an open interval I containing b such that $f(x) > 0$ for all $x \in I$.

Problem 4. Consider the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots \quad . \tag{1}$$

Clearly, this sequence can be defined recursively, i.e., by

$$a_{n+1} = f(a_n)$$

for some appropriately chosen function f.

(a) Prove by induction that the sequence (1) is increasing.

(b) Prove by induction that the sequence (1) is bounded.

(c) Prove that the sequence (1) converges.

(d) Find the limit of the sequence (1).

Problem 5. Let (a_n) be a bounded sequence.

(a) Prove that the sequence (y_n) defined by $y_n = \sup\{a_k : k \geq n\}$ converges.
The *limit superior* of (a_n), or $\limsup a_n$, is defined by $\limsup a_n := \lim y_n$.

(b) Prove that the sequence (z_n) defined by $z_n = \inf\{a_k : k \geq n\}$ converges.
The *limit inferieor* of (a_n), or $\liminf a_n$, is defined by $\liminf a_n := \lim z_n$.

(c) Prove that $\liminf a_n \leq \limsup a_n$.

Problem 6. Prove that if $\sum_{k=0}^{\infty} a_k$ is a convergent series, then $\lim_{n\to\infty} a_n = 0$.

Problem 7. Consider the infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$.

(a) Find constants α and β such that

$$\frac{1}{k(k+2)} = \frac{\alpha}{k} + \frac{\beta}{k+2} .$$

(b) Show by induction that the nth partial sum of the series is

$$\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{n(3n+5)}{4(n+1)(n+2)}$$

(this result is inspired by using the result of part (a); keyword: *telescoping*).

(c) Find the sum of $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$.

Problem 8.

(a) Use the basic properties of logarithms to write the series

$$\sum_{k=1}^{\infty} \ln \sqrt{\frac{k}{k+1}}$$

as a telescoping series, and find an explicit expression for its partial sums. Does the series converge?

(b) Use the fact that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ to find the sums of the series

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots$$

and

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$