MATH 4093/5093 **Homework 1** **Due Thu, 09/02/10**

Problem 1. The following theorem is an important result in Calculus.

> **Intermediate Value Theorem (IVT).** If f is continuous on $[a, b]$ and k is *any* number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ satisfying $f(c) = k$.

Use this theorem to prove that the equation $(x - 2)^2 = \ln x$ has at least one solution in the interval $[1, 2]$.

Hint: Write the equation in the form $f(x) = 0$ for an appropriately defined function $f(x)$, and apply the Intermediate Value Theorem.

Problem 2. Consider the function

$$f(x) = \mathrm{e}^{2x/\pi} + (1 - \mathrm{e}) \sin x \ ,$$

where $\mathrm{e} = 2.718281828459\ldots$ is the base of the natural logarithms.

(a) Use the Mean Value Theorem to show that the derivative of f vanishes (i.e., becomes equal to zero) at least once in the interval $[0, \frac{\pi}{2}]$, without computing f' explicitly.

Hint: Find the values of $f(0)$ and $f(\frac{\pi}{2})$ and use them.

(b) In the rest of this problem you will give another solution of what you already proved in part (a), and, in addition, will show that the point where f' vanishes is unique. Start by finding the derivative of f explicitly.

(c) Apply the Intermediate Value Theorem from the statement of Problem 1 above to the equation $f'(x) = 0$ that you have derived in part (b) to prove that the equation $f'(x) = 0$ has at least one solution in the interval $[0, \frac{\pi}{2}]$.

(d) Show that the solution of $f'(x) = 0$ whose existence was proved in part (c) is in fact unique.

Hint: If you take the derivative of the left-hand side of the equation $f'(x) = 0$, you will obtain

$$f''(x) = \frac{4}{\pi^2} \mathrm{e}^{2x/\pi} + (\mathrm{e} - 1) \sin x \ .$$

What can you say about the sign of $f''(x)$ for $x \in [0, \frac{\pi}{2}]$? What does this imply for the behavior of $f'(x)$ on this interval?

Problem 3. Applying the L'Hospital rule to find limits of ratios where both the numerator and the denominator tend to zero is sometimes long and error-prone. Use the Taylor expansions

$$\mathrm{e}^{x^2} = 1 + \frac{1}{1!}(x^2) + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \cdots \ , \qquad \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \ ,$$

to compute the value of the limit

$$\lim_{x \to 0} \frac{\exp(x^2) - \cos x}{x^2} \ .$$

Problem 4. In this problem you will use Taylor's Theorem to approximate the value of $\sqrt{17}$.

(a) Write the second-degree Taylor polynomial,

$$P_2(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2$$

for the function $f(x) = \sqrt{x}$ around $x_0 = 16$. You have to find explicitly the numerical values of the coefficients of $P_2(x)$; there is no need to expand the factors $(x-x_0)^j$.

Hint: The answer is: $P_2(x) = 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2$, but I want to see your derivations.

(b) Find the numerical value of $P_2(17)$.

(c) Show that the remainder term is

$$R_2(x) = \frac{f'''(\xi(x))}{3!}(x-x_0)^3 = \frac{1}{16\,[\xi(x)]^{5/2}}\left(x-16\right)^3 ,$$

and find the maximum possible value of $|R_2(17)|$. Here $\xi(x)$ is a number between $x_0 = 16$ and $x = 17$; this number is unknown, so to find the maximum possible value of $|R_2(17)|$, you have to allow $\xi(x)$ to be *anywhere* between 16 and 17. The maximum possible value of $|R_2(17)|$ is a *rigorous* upper bound on the size of the error if you replace the exact value $f(17) = \sqrt{17}$ with its approximation, $P_2(17)$.

(d) Compute the true numerical value of the so-called *absolute error*, $|P_2(17) - \sqrt{17}|$ (the absolute error is the absolute value of the difference between the exact and the approximate values). Compare the true value of $|P_2(17) - \sqrt{17}|$ with the upper bound for the error obtained in part (c). Discuss briefly your observations.

Problem 5. In this problem you will show that some functions are Lipschitz and will find their Lipschitz constants.

(a) Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(t,y) = y\cos t$. Show that f is Lipschitz in y on $\mathbb{R}^2$, and the value of the Lipschitz constant is $L = 1$.

Hint: This example is practically the same as the first part of Example 7.3 on page 542 of the book.

(b) Let $D = \left\{(t,y) \in \mathbb{R}^2 \,:\, 0 \le t \le 3\,,\, 0 \le y \le \frac{\pi}{4}\right\}$, and let $f : D \to \mathbb{R} : (t,y) \mapsto \frac{t}{t+1}\tan y$. Show that f is Lipschits in y on D and find the value of the Lipschits constant.

Hint: Apply the same ideas as in Example 7.4 on page 544 of the book, in which the theorem on page 543 is used to compute the Lipschitz constant of a function.

Problem 6. Solve parts (a)-(d) of Exercise 1 on page 544 of the book. One-two sentences of explanation for each part is enough.