MATH 5163 Homework 1 Due Thu, 1/23/14

Problem 1. Consider the first-order PDE

$$x u_x + y u u_y = u \tag{1}$$

for the unknown function $u(x,y)$, in the first quadrant, i.e., in the domain

$$U := \{(x,y) \in \mathbb{R}^2 : x > 0,\ y > 0\}\ .$$

(a) Prove that, for any smooth function ϕ of one variable, the function $u(x,y)$ defined implicitly by the relation

$$u = \ln y + \phi\left(\frac{u}{x}\right) \tag{2}$$

satisfies the PDE (1).

Hint: The easiest way to do this is to write (2) as

$$u(x,y) = \ln y + \phi\left(\frac{u(x,y)}{x}\right)\ ,$$

and to differentiate this relation with respect to x and y in order to obtain explicit expressions for u_x and u_y (which, of course, will contain derivatives of the function ϕ), and then show that these expressions satisfy the PDE (1).

Remark: Note that the form of the relation (2) is consistent with the fact mentioned in class that the general solution of an order-k PDE for an unknown function u of m variables contains k arbitrary functions of $(m-1)$ variables.

(b) Now, in addition to the PDE (1), let us impose on u the condition that, on the part of the curve $y = \mathrm{e}^x$ that lies in the 1st quadrant – which can be written in a parametric form as $\{(x(t), y(t)) = (t, \mathrm{e}^t) : t > 0\}$ – the value of the function u is given by

$$u(t, \mathrm{e}^t) = \frac{t^2}{t+1}\ . \tag{3}$$

The solution of the 1st order PDE (2) that also satisfies the condition (3) is

$$u(x,y) = \frac{x \ln y}{x+1}\ . \tag{4}$$

Check that the function (4) satisfies the condition (3).

(c) The function $u(x,y)$ defined in (4) belongs to the family of functions defined by (2). What is the particular form of the function ϕ for which the function u defined implicitly by (2) takes the form (4)?

Problem 2. Find the general solution of the partial differential equation

$$u_{xxy} = \frac{2y}{x}$$

for the function $u(x, y)$. Do not forget that your answer should contain a certain number of arbitrary functions!

Problem 3. In class we wrote the equation describing the motion of a membrane in the Earth's gravity field,

$$\rho\, u_{tt} = \tau\, \Delta u - \rho g \ . \tag{5}$$

Here $u(x, y, t)$ is the function that describes the position of the membrane at time t by $z = u(x, y, t)$, where $(x, y) \in U \subset \mathbb{R}^2$; ρ is the density of the membrane (i.e., the mass per unit area), τ is the surface tension, and $g = 9.8\,\mathrm{m/s}^2$ is the free-fall acceleration.

In this problem you will study the equilibrium shape of a circular membrane acted upon by the gravity field. Since the equilibrium shape of the membrane does not depend on the time, the unknown function u become a function only of x and y, i.e., the equilibrium shape of the membrane is given by $z = u(x, y)$ (we use the same letter for the unknown function as above although, clearly, it represents a different function). The function $u(x, y)$ satisfies the Poisson equation

$$\Delta u = \frac{\rho g}{\tau} \ , \tag{6}$$

which follows directly from (5) by setting all time derivatives equal to zero.

In the steps below you will find the shape of a membrane attached to a wire with a shape of a circle of radius a, i.e., you will solve the boundary value problem

$$\begin{aligned} &\Delta u = \frac{\rho g}{\tau} \ , \qquad (x, y) \in U := \{(x, y) : x^2 + y^2 \le a^2\} \ , \\ &u|_{\partial U} = 0 \ . \end{aligned} \tag{7}$$

Because of the symmetry of the problem, it is clear that, if we use polar coordinates (r, θ) in the (x, y)-plane, the function describing the shape of the membrane will depend only on r. Therefore, in polar coordinates the Poisson equation in (7) will become an ordinary differential equation. Below you will first compute the Laplacian Δ in polar coordinates, and then will find the equilibrium shape of the membrane.

The polar coordinates (r, θ) in $\mathbb{R}^2$ are defined in terms of the Cartesian coordinates (x, y) by

$$\begin{aligned} x &= X(r, \theta) := r \cos\theta \ , \\ y &= Y(r, \theta) := r \sin\theta \ , \end{aligned} \tag{8}$$

and the inverse change is given by

$$r = R(x,y) = \sqrt{x^2+y^2}\ ,$$

$$\theta = \Theta(x,y) = \begin{cases} \arctan\frac{y}{x}\ , & x>0 \\ \pi + \arctan\frac{y}{x}\ , & x<0 \\ \frac{\pi}{2}\ , & x=0\,,\ y>0 \\ \frac{3\pi}{2}\ , & x=0\,,\ y<0. \end{cases} \tag{9}$$

Despite the complicated form of the function $\Theta(x,y)$, its geometric meaning is very simple – it is the angle between the positive x-direction and the line connecting the origin of $\mathbb{R}^2$ with the point (x,y) (in counterclockwise direction). Here the capital letters X, Y, R, and Θ are used to denote functions, while x, y, r, and θ stand for variables.

Now we define the function $\widetilde{u}(r,\theta)$, which is equal to the function $u(x,y)$ if polar coordinates are used. By definition,

$$\widetilde{u}(r,\theta) := u(x,y)\Big|_{x=X(r,\theta),\ y=Y(r,\theta)} = u(X(r,\theta),Y(r,\theta))\ ,$$

where the functions X and Y are as in (8). This can also be written as

$$u(x,y) = \widetilde{u}(R(x,y),\Theta(x,y))\ . \tag{10}$$

with R and Θ given by (9).

Here are some results that you will need in your calculations:

$$\frac{\partial R}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}\ ,\quad \frac{\partial R}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}\ ,\quad \frac{\partial \Theta}{\partial x} = -\frac{y}{x^2+y^2}\ ,\quad \frac{\partial \Theta}{\partial y} = \frac{x}{x^2+y^2}\ .$$

$$\frac{\partial^2 R}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{3/2}}\ ,\quad \frac{\partial^2 R}{\partial y^2} = \frac{x^2}{(x^2+y^2)^{3/2}}\ ,\quad \frac{\partial^2 \Theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}\ ,\quad \frac{\partial^2 \Theta}{\partial y^2} = -\frac{2xy}{(x^2+y^2)^2}\ .$$

(a) Express u_{xx} and u_{yy} in terms of the function $\widetilde{u}$ and its derivatives. You have to use (10) and the chain rule as follows:

$$u_x(x,y) = \frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial x}\widetilde{u}(R(x,y),\Theta(x,y)) = \frac{\partial \widetilde{u}}{\partial r}\frac{\partial R}{\partial x} + \frac{\partial \widetilde{u}}{\partial \theta}\frac{\partial \Theta}{\partial x}\ ,$$

and for the second x-derivative,

$$
\begin{aligned}
u_{xx}(x,y) &= \frac{\partial}{\partial x}\, u_x(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial r}\,\frac{\partial R}{\partial x} + \frac{\partial \widetilde{u}}{\partial \theta}\,\frac{\partial \Theta}{\partial x}\right) \\
&= \frac{\partial^2 \widetilde{u}}{\partial r^2}\left(\frac{\partial R}{\partial x}\right)^2 + \frac{\partial^2 \widetilde{u}}{\partial r\,\partial \theta}\,\frac{\partial R}{\partial x}\,\frac{\partial \Theta}{\partial x} + \frac{\partial \widetilde{u}}{\partial r}\,\frac{\partial^2 R}{\partial x^2} \\
&\quad + \frac{\partial^2 \widetilde{u}}{\partial r\,\partial \theta}\,\frac{\partial R}{\partial x}\,\frac{\partial \Theta}{\partial x} + \frac{\partial^2 \widetilde{u}}{\partial \theta^2}\left(\frac{\partial \Theta}{\partial x}\right)^2 + \frac{\partial \widetilde{u}}{\partial \theta}\,\frac{\partial^2 \Theta}{\partial x^2} \\
&= \widetilde{u}_{rr}\,\frac{x^2}{x^2+y^2} + \widetilde{u}_{r\theta}\,\frac{x}{\sqrt{x^2+y^2}}\left(-\frac{y}{x^2+y^2}\right) + \widetilde{u}_r\,\frac{y^2}{(x^2+y^2)^{3/2}} \\
&\quad + \widetilde{u}_{r\theta}\,\frac{x}{\sqrt{x^2+y^2}}\left(-\frac{y}{x^2+y^2}\right) + \widetilde{u}_{\theta\theta}\left(-\frac{y}{x^2+y^2}\right)^2 + \widetilde{u}_\theta\,\frac{2xy}{(x^2+y^2)^2} \\
&= \widetilde{u}_{rr}\,\frac{x^2}{x^2+y^2} - \widetilde{u}_{r\theta}\,\frac{2xy}{(x^2+y^2)^{3/2}} + \widetilde{u}_r\,\frac{y^2}{(x^2+y^2)^{3/2}} \\
&\quad + \widetilde{u}_{\theta\theta}\,\frac{y^2}{(x^2+y^2)^2} + \widetilde{u}_\theta\,\frac{2xy}{(x^2+y^2)^2}\;.
\end{aligned}
$$

Write down the calculations for u_y and u_{yy} in detail.

(b) Use your result from (a) to show that in polar coordinates the Laplacian is given by

$$
\Delta\widetilde{u} = \widetilde{u}_{rr} + \frac{1}{r}\,\widetilde{u}_r + \frac{1}{r^2}\,\widetilde{u}_{\theta\theta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \widetilde{u}}{\partial r}\right) + \frac{1}{r^2}\,\widetilde{u}_{\theta\theta}\;.
$$

(c) Since for a circular membrane the coordinate θ is immaterial, set $\widetilde{v}(r) := \widetilde{u}(r,\theta)$, and rewrite the boundary value problem (7) as a boundary value problem for $\widetilde{v}(r)$:

$$
\begin{aligned}
&\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}\widetilde{v}}{\mathrm{d}r}\right) = \frac{\rho g}{\tau}\;, \qquad r \in [0,a]\;, \\
&|\widetilde{v}(0)| < \infty\;, \qquad \widetilde{v}(a) = 0\;.
\end{aligned}
\tag{11}
$$

Explain how this boundary value problem was obtained; in particular, where did the boundary conditions come from?

(d) Solve the boundary value problem (11) explicitly to find the equilibrium shape of the membrane. Please explain clearly how you apply the boundary conditions to find the two arbitrary constants coming from the integration.

(e) How far is the lowest point of the membrane from the (x,y)-plane?