MATH 5403 Homework 1 Due Thu, 09/26/19

Problem 1. [A useful lemma]
Directly from the definition of continuity, prove that if $f \in C[a,b]$, $f \geq 0$ on $[a,b]$, and $\int_a^b f(x)\,\mathrm{d}x = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

Problem 2. [Derivation of the Euler-Lagrange equation]

(a) Write down the Euler-Lagrange equation for the action functional

$$J[y] = \int_a^b (y \sin y' + xy')\,\mathrm{d}x\ .$$

Expand all terms completely, do not leave any parentheses.

(b) Write down the Euler-Lagrange equation for the action functional

$$J[y] = \int_a^b \left[x(y')^2 - yy' + y\right]\mathrm{d}x\ .$$

Again, expand all terms completely.

Problem 3. [A particular solution of the Euler-Lagrange equation]
Consider the action functional

$$J[y] = \int_a^b y'\left(1 + x^2 y'\right)\mathrm{d}x\ . \tag{1}$$

(a) Derive the Euler-Lagrange equation for the action functional $J[y]$ given by (1).

(b) Use that $\dfrac{\mathrm{d}}{\mathrm{d}x}\,g(x) = 0$ implies $g(x) = \text{const}$ to reduce the Euler-Lagrange equation from part (a) to a first-order ODE; it will be a simple separable ODE.

(c) Find the general solution of the separable ODE derived in part (b) (I want to see your calculations). Show that the general solution you found can be written in the form $y(x) = \dfrac{C_1}{x} + C_2$, where C_1 and C_2 are arbitrary constants.

(d) Impose the boundary conditions $y(1) = 5$, $y(2) = 3$ on the solution $y(x)$ on the interval $x \in [1,2]$ to find the particular solution of the Euler-Lagrange equation for the action functional (1) derived in part (c).

(e) Denote the particular solution obtained in part (d) by $\hat{y}(x)$. Find the numerical value of the action integral

$$I[\hat{y}] = \int_1^2 \hat{y}\,'(x)\left[1 + x^2\hat{y}\,'(x)\right]\mathrm{d}x \ .$$

for the particular choice of function $\hat{y}(x)$, and for $a = 1$, $b = 2$.

Problem 4. [Energy change along a solution of the Euler-Lagrange equation]

(a) Consider a particle moving in one dimension, whose position at time t is $q(t)$. Let the action for this system be $J[q] = \displaystyle\int_a^b L(t, q, \dot{q})\,\mathrm{d}t$, where the Lagrangian is

$$L(t, q, \dot{q}, t) = \dot{q}^2 - q^2 + 2q\sin t \ , \tag{2}$$

and $\dot{q} := \dfrac{\mathrm{d}q}{\mathrm{d}t}$. Write down the Euler-Lagrange equation of this system. (The equation you will obtain is the equation describing a periodically forced harmonic oscillator.)

(b) Find the general solution of the Euler-Lagrange equation derived in part (a). The equation you have to solve is a non-homogeneous linear second order constant-coefficient ODE of the form $Dq(t) = f(t)$, where D is a differential operator (see (4) below), and $f(t)$ is a given function. The general solution $q(t)$ of such an equation is equal to the sum of the general solution $q_{\text{gen,hom}}(t)$ of the homogeneous equation $Dq(t) = 0$ and a particular solution $q_{\text{part,nonhom}}(t)$ of the non-homogeneous equation $Dq(t) = f(t)$:

$$q(t) = q_{\text{gen,hom}}(t) + q_{\text{part,nonhom}}(t) \ .$$

The methods for finding $q_{\text{gen,hom}}(t)$ and $q_{\text{part,nonhom}}(t)$ are described in the Appendix. For your convenience, here you may use that in this problem $q_{\text{part,nonhom}}(t) = -\dfrac{t}{2}\sin t$, without deriving it.

(c) The system described by the Lagrangian (2) is clearly not autonomous, so its energy

$$E(t) = q'(t)\,\frac{\partial L}{\partial q'} - L$$

will depend on time. Show that the energy E of this system can be written as $E = \dot{q}^2 + q^2 - 2q\sin t$.

(d) Take time derivative of the energy obtained in part (c), and use that along a true trajectory $\hat{q}(t)$ the Euler-Lagrange equation holds, in order to obtain that the energy changes with time as $\dfrac{\mathrm{d}E}{\mathrm{d}t} = -2q(t)\cos t$. Do this without using the expression for the function $q(t)$ obtained in part (b).

Problem 5. [Euler-Lagrange equations for two functions of one variable]
Consider a curve in the plane defined by the parametric equations

$$x = q_1(t) \ , \qquad y = q_2(t) \ , \tag{3}$$

where $q_1(t)$ and $q_2(t)$ are some functions of a parameter t.

(a) The functional $J[q_1, q_2] = \int_a^b \sqrt{\dot{q}_1^2 + \dot{q}_2^2}\, dt$ describes the length of the curve (3) between the points $(q_1(a), q_2(a))$ and $(q_1(b), q_2(b))$. Write down the Euler-Lagrange equations for the action $J[q_1, q_2]$.

(b) What we really want to find in this problem is q_2 as a function of q_1. Divide the first equation obtained in part (a) by the second one, and use the Chain Rule to obtain an ODE for the function $q_2(q_1)$. Explain specifically where and how you used the Chain rule.

(c) Find the general solution of the ODE for $q_2(q_1)$ derived in part (b).

(d) Your result from part (c) has a very simple geometric meaning – what is it?

APPENDIX

General solution of a linear homogeneous ODE with constant coefficients:

To find the general solution of the linear homogeneous ordinary differential equation with constant coefficients $Dy = 0$, where

$$Dy := a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0\, y \ , \tag{4}$$

first solve the characteristic equation of this ODE. Each root of the characteristic equation contributes a term to the general solution of the ODE:

- each real root r of the characteristic equation of multiplicity p contributes to the general solution a term
$$P_{p-1}(x)\, e^{rx} \ ,$$
where $P_{p-1}(x)$ is a polynomial of degree $(p-1)$;

- each conjugate pair of complex roots $\alpha \pm i\beta$ of the characteristic equation, where each of the the two roots has multiplicity p, contributes to the general solution a term
$$e^{\alpha x}\left[Q_{p-1}(x)\cos\beta x + R_{p-1}(x)\sin\beta x\right] \ ,$$
where $Q_{p-1}(x)$ and $R_{p-1}(x)$ are polynomials of degree $(p-1)$.

Particular solutions of a linear non-homogeneous ODE with constant coefficients

The general solution of the linear non-homogeneous ordinary differential equation with constant coefficients $Dy = f(x)$ (where Dy is given by the expression (4) above) is equal to the sum of the general solution of the associated homogeneous equation $Dy = 0$ and a particular solution of $Dy = f(x)$. First solve the characteristic equation of $Dy = 0$ and find the general solution of $Dy = 0$, and then find a particular solution of $Dy = f(x)$ by doing the following:

- in the case $f(x) = e^{cx}\, P_m(x)$, if c is a root of the characteristic equation of $Dy = 0$ with multiplicity s, then look for a particular solution $y_p(x)$ of $Dy = f(x)$ of the form

  $$y_p(x) = x^s\, e^{cx}\, Q_m(x) \ ,$$

  where $Q_m(x)$ is a polynomial of degree m;

- in the case $f(x) = e^{cx}\left[P_{m_1}(x)\cos dx + \widetilde{P}_{m_2}(x)\sin dx\right]$, if $c + id$ is a root of the characteristic equation of $Dy = 0$ with multiplicity s, then define $m := \max(m_1, m_2)$, and look for a particular solution $y_p(x)$ of $Dy = f(x)$ of the form

  $$y_p(x) = x^s\, e^{cx}\left[Q_m(x)\cos dx + \widetilde{Q}_m(x)\sin dx\right] \ ,$$

  where $Q_m(x)$ and $\widetilde{Q}_m(x)$ are polynomials of degree m.

If the equation has the form $Dy = f_1(x) + f_2(x)$, its general solution is a sum of the general solution of $Dy = 0$ and the particular solutions of the equations $Dy = f_1(x)$ and $Dy = f_2(x)$.