MATH 5463 Homework 1 Due Thu, Feb 5

Problems 22, 23, 24 from Section 3.4 of the book.

Additional problem 1.

(a) Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a jointly continuous function of its arguments, then for each fixed $y \in \mathbb{R}$ the function $g_y : \mathbb{R} \to \mathbb{R}$ defined by $g_y(x) := f(x, y)$ is continuous.

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\ , \\ 0 & \text{if } (x,y) = (0,0)\ . \end{cases}$$

Show that f is continuous in each of its arguments for any fixed value of the other argument, but is *not* jointly continuous in $\mathbb{R}^2$. Moreover, f cannot be made jointly continuous even if we were able to change its value at $(0,0)$ to any value.

Additional problem 2. The functions $S_n : (-1,1) \to \mathbb{R}$ defined by

$$S_n(x) = 1 + \sum_{k=0}^{n} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k\, k!}\, x^k$$

are the partial sums of the Maclaurin series representing the function $f : x \mapsto \dfrac{1}{\sqrt{1-x}}$ defined for $x \in (-1,1)$. Prove that S_n converge in $L^1((-1,1), m)$ to f.

Hint: Prove that, for each $x \in (-1,1)$, the sequence $\{S_n(|x|)\}_{n=1}^{\infty}$ is monotone increasing, then show that, for the same range of x, $0 \le S_n(x) \le S_n(|x|) \le f(|x|)$, use that the function $x \mapsto \dfrac{1}{\sqrt{1-|x|}}$ is in $L^1((-1,1), m)$ and apply some convergence theorem.

Additional problem 3. Let

$$\Theta(x) = \begin{cases} 0 & \text{if } x < 0\ , \\ c & \text{if } x = 0\ , \\ 1 & \text{if } x > 0\ , \end{cases}$$

where $c \in [0,1]$ is a constant which you are free to change as you wish. Is there a value for c for which the Lebesgue set of the function Θ will include the point $x = 0$?

Food for thought.[1] Show that $\lim_{r \to R} \phi(r) = c$ is equivalent to $\limsup_{r \to R} |\phi(r) - c|$.

[1] "Food for thought" problems are not to be turned in. They are just for you to learn some facts and think about their proofs.

Food for thought. Let (x, y) and (r, θ) be the Cartesian and the polar coordinates in the plane $\mathbb{R}^2$, respectively, and let $\mathbf{0}$ be the origin, $(x, y) = (0, 0)$. For $n \in \mathbb{N}$ define the numbers

$$a_n = \frac{1}{10^n}\left(1 - \frac{1}{10^n}\right) , \qquad b_n = \frac{1}{10^n}$$

and the sets $D_n = \{(x, y) \in \mathbb{R}^2 : r \in [a_n, b_n)\}$. Each D_n is an *annulus* (plural: *annuli*), i.e., the area between two concentric circles: $D_n = B(b_n, \mathbf{0}) \setminus B(a_n, \mathbf{0})$. Note that D_n are disjoint. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x, y) = \sum_{n \in \mathbb{N}} \chi_{D_n}(x, y) ,$$

i.e., $f(x, y)$ is 1 if (x, y) belongs to some annulus D_n and 0 otherwise. Note that $f(\mathbf{0}) = 0$.

(a) Is the function f continuous at $\mathbf{0}$?

(b) Show that the Lebesgue measure (i.e., the area) of D_n is smaller than $\frac{2\pi}{10^{3n}}$.

(c) Let $(x, y) \in D_n$, i.e., $r \in [a_n, b_n)$. Prove that the average $(A_r f)(\mathbf{0})$ of f over the ball $B(r, \mathbf{0})$ decreases with n as $\frac{C}{10^n}$ for some constant $C > 0$.

Solution:

$$\begin{aligned}
(A_r f)(\mathbf{0}) &= \frac{1}{m(B(r, \mathbf{0}))} \int_{B(r,\mathbf{0})} f \, dm \;\le\; \frac{1}{m(B(a_n, \mathbf{0}))} \int_{B(b_n,\mathbf{0})} f \, dm \\
&= \frac{1}{\pi a_n^2} \sum_{j=n}^{\infty} m(D_n) \;<\; \frac{10^{2n}}{\pi\left(1 - \frac{1}{10^{2n}}\right)^2} \sum_{j=n}^{\infty} \frac{2\pi}{10^{3j}} \\
&\le \frac{2 \cdot 10^{2n}}{\pi} \frac{2\pi}{10^{3n}} \sum_{k=0}^{\infty} \frac{1}{10^{3k}} \;=\; \frac{C}{10^n} .
\end{aligned}$$

(d) Let (x, y) be a point in the area between D_n and D_{n+1}, i.e, let the distance from (x, y) to $\mathbf{0}$ be $r \in [b_{n+1}, a_n)$. Give an upper bound on the average $(A_r f)(\mathbf{0})$ of f over the ball $B(r, \mathbf{0})$ in terms of n similarly to the bound in part (c).

(e) Based on the bounds in parts (c) and (d), what can you conclude about the behavior of the averages $(A_r f)(\mathbf{0})$ as $r \to 0$? How about the Hardy-Littlewood maximal function $(Hf)(\mathbf{0})$? Is the point $\mathbf{0}$ in the Lebesgue set of f?