

Problem 1. Let the sample space S be equal to the interval $(0, 1]$, and its sets A and B be defined as

$$A := \left(0, \frac{1}{3}\right] , \quad B := \left(0, \frac{1}{2}\right] .$$

Write down explicitly (as subsets of $(0, 1]$) all elements of the smallest σ -field \mathcal{F} of subsets of S that contains A and B . Explain in a couple of sentences how you obtained \mathcal{F} .

Hint: The desired σ -field consists of exactly 8 subsets (including the empty set and S).

Problem 2. Let \mathcal{F} be a σ -algebra of subsets of the sample space S , and let $B \in \mathcal{F}$. Consider the collection

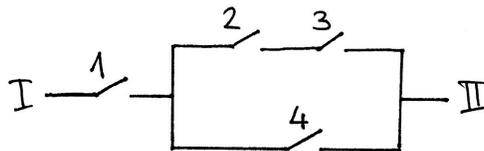
$$\mathcal{G} := \{A \cap B : A \in \mathcal{F}\}$$

of subsets of B . Show that \mathcal{G} is a σ -algebra of subsets of B .

Problem 3. Let A and B be events with $\mathbb{P}(A) = \frac{3}{4}$, and $\mathbb{P}(B) = \frac{1}{3}$.

- (a) Show that $\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}$, and draw pictures (or give a brief verbal explanation) to illustrate that both extremes are possible.
- (b) Find the corresponding bounds for $\mathbb{P}(A \cup B)$.

Problem 4. The electric circuit shown in the figure below contains four switches. Switch $\#i$ is closed (i.e., letting the electric current through) with probability p_i . The switches function independently, i.e.e, the state of switch $\#i$ does not depend on the states of switches $\#k$ for $k \neq i$. Let E_i denotes the event that switch $\#i$ is closed.



- (a) Express the event $C :=$ “the electric current flows from I to II” in terms of the events E_i ($i = 1, 2, 3, 4$) by using the operations union, intersection, and/or complement.
- (b) Use your result from part (a) to write the probability $\mathbb{P}(C)$ in terms of the probabilities p_i ($i = 1, 2, 3, 4$). Here you should just use your result from (a), *without conditioning!*
- (c) Find the conditional probability of the event C given that switch $\#4$ is closed (i.e., conditioned on the event E_4).

- (d) Determine the conditional probability of the event C given that the switch #4 is open.
- (e) Use your results from parts (c) and (d) to express $\mathbb{P}(C)$ in terms of the probabilities p_i ($i = 1, 2, 3, 4$).
- (f) Given that the event C occurs, what is the probability that switch #4 is closed?

Problem 5. Scott and Curtis go target shooting together. Both shoot at a target at the same time. Suppose that Scott hits the target with probability $p_S = 0.7$, whereas Curtis, independently, hits the target with probability $p_C = 0.4$.

- (a) Determine the probability that the target is hit.
- (b) Given that the target is hit, what is the probability that Curtis hit it?
- (c) Find the probability that *exactly* one shot hits the target.
- (d) Given that *exactly* one shot hits the target, what is the probability that it was Curtis's shot?

Problem 6. Suppose that two fair dice – a red one and a green one – are rolled. Consider the events

$$\begin{aligned}
 A &= \text{“The red die shows an odd number.”} \\
 B &= \text{“The green die shows an odd number.”} \\
 C &= \text{“The sum of the numbers on the two dice is odd.”}
 \end{aligned}$$

Show that the events A , B , and C are pairwise independent, but not independent.

Problem 7. A pair of dice is rolled until a sum of either 5 or 7 appears. In this problem we will find the probability that a sum equal to 5 occurs first. Please follow the steps below.

- (a) What is the probability that *in one individual roll of the two dice* the sum will be 5? For your convenience, the table below represents all possible outcomes. Think of the two dice as being distinct (say, one of them is red and the other is green). Then the first number in each pair represents the outcome of the red die, and the second one represents the outcome of the green die.

| | | | | | |
|--------|--------|--------|--------|--------|--------|
| (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) |
| (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) |
| (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) |
| (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) |
| (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) |
| (6, 1) | (6, 2) | (6, 3) | (6, 4) | (6, 5) | (6, 6) |

- (b) What is the probability that *in one individual roll of the two dice* the sum will be neither 5 nor 7?
- (c) Let E_n be the event that in the sequence of rolls a 5 occurs *for the first time* on the n th roll, and no 7 has occurred before that. (In other words, E_n is the event that a 5 occurs on the n th roll and no 5 or 7 occurs in the first $n - 1$ rolls.) Find the probability $P(E_n)$ of the event E_n .
- (d) Argue that the desired probability (i.e., the probability that a 5 occurs first) is equal to the infinite sum $\sum_{n=1}^{\infty} P(E_n)$.
- (e) Find the value of the desired probability by calculating the above sum. You may need the formula for the sum of a geometric series, $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, valid whenever $|q| < 1$.