MATH 3413 Homework 10 Due Thu, Nov 9

Sec. 9.1: problems 15, 17.

Additional problem 1. Let $g(t)$ be a periodic function of period 2π defined by

$$g(t) = \begin{cases} 0 & \text{if } -\pi < t < 0, \\ 6 & \text{if } 0 < t < \pi \ ; \end{cases}$$

the values of $g(t)$ at the points $x = 0$ and $x = \pm\pi$ are immaterial.

(a) Express the function $g(t)$ in terms of the "square wave" function $f(t)$ defined in Example 1 of Sec. 9.1 of the book.

(b) Use your result from part (a) to write down the Fourier series of $g(t)$; assume that you know that the Fourier series of the "square wave" function is

$$f(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n} = \frac{4}{\pi} \left(\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots \right) .$$

Additional problem 2. Each function $f(t)$ can be written as a sum of an even and an odd function as follows:

$$f(t) = f_{\text{even}}(t) + f_{\text{odd}}(t) := \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2} .$$

(a) Show that indeed the functions

$$f_{\text{even}}(t) = \frac{f(t) + f(-t)}{2} , \quad \text{and} \quad f_{\text{odd}}(t) = \frac{f(t) - f(-t)}{2}$$

are even and odd, respectively.

(b) Show that the even-odd decomposition of the function $\ln|1-t|$ is

$$\ln|1-t| = (\ln|1-t|)_{\text{even}} + (\ln|1-t|)_{\text{odd}} = \ln\sqrt{|1-t^2|} + \ln\sqrt{\left|\frac{1-t}{1+t}\right|} .$$

(c) Find the even-odd decomposition of the functions e^t, $1 - x^3 - 2x^4 + 7x^5$, and $\sin^2 x$.

(d) Show that the product of an odd and an even function is odd, the product of two odd functions is even, and the product of two even functions is even. Use this to find the even-odd decomposition of the function $(1+x)(\cos x + \sin x)$.

Additional problem 3. Find the period and the Fourier series of the function $\cos^2(5x)$.

Hint: The easiest way to write this Fourier series is to use trigonometric identities.

Additional problem 4.

(a) Set $t = \frac{\pi}{2}$ in the Fourier series you obtained in problem 15 of Sec. 9.1. What identity do you obtain? (Sometimes this identity is called *Leibniz's series.*)

(b) Set $t = \pi$ in the Fourier series you obtained in problem 17 of Sec. 9.1. What identity do you obtain?

Additional problem 5. Let $g(t)$ be the function from problem 17 of Sec. 9.1, and $\mathrm{FS}(t)$ be its Fourier series.

(a) Take derivatives (with respect to t) of both sides of the "equality" $g(t) \sim \mathrm{FS}(t)$ (the word "equality" is in quotation marks because $g(t)$ and $\mathrm{FS}(t)$ can differ for some values of t, as explained in Theorem 1 of Sec. 9.2).

(b) Compare the series $\frac{d}{dt}\,\mathrm{FS}(t)$ with the Fourier series of the "square wave" function from Example 1 of Sec. 9.1. Hmm, can this be just a coincidence? Can you suggest some explanation?

Additional problem 6. A thin wooden rod is attached to the point with coordinates $(0,1)$ in the (x,y)-plane, and it is clamped at this point, so that it starts off in positive x-direction. The other end of the rod passes under a thin peg at the point $(10,0)$. The rod is shown in Figure 1. Let the function $f(x)$ give the shape of the rod: $y = f(x)$. One can show that the bending of a rod is governed by the fourth order ordinary differential equation

$$f^{(4)}(x) = 0 \ . \tag{1}$$

According to the description above, the function $f(x)$ must satisfy the following *boundary* conditions (not only *initial* conditions at $x = 0$, but also *final* conditions at $x = 10$):

$$\begin{aligned}
f(0) &= 1 \quad \text{[the rod passes through the point } (0,1)] \ , \\
f'(0) &= 0 \quad \text{[the rod is clamped at } (0,1) \text{ and "starts off" horizontally]} \ , \\
f(10) &= 0 \quad \text{[the rod passes through the point } (10,0)] \ , \\
f''(10) &= 0 \quad \text{[the right end of the rod is free (no bending forces)]} \ .
\end{aligned}$$

(a) Write down the general solution of the differential equation (1). Since this is a fourth order differential equation, its general solution must contain four arbitrary constants C_1, C_2, C_3, C_4.

(b) Impose the boundary conditions to find the solution of the *boundary value problem*

$$f^{(4)}(x) = 0 \ , \quad f(0) = 1 \ , \quad f'(0) = f(10) = f''(10) = 0 \ .$$

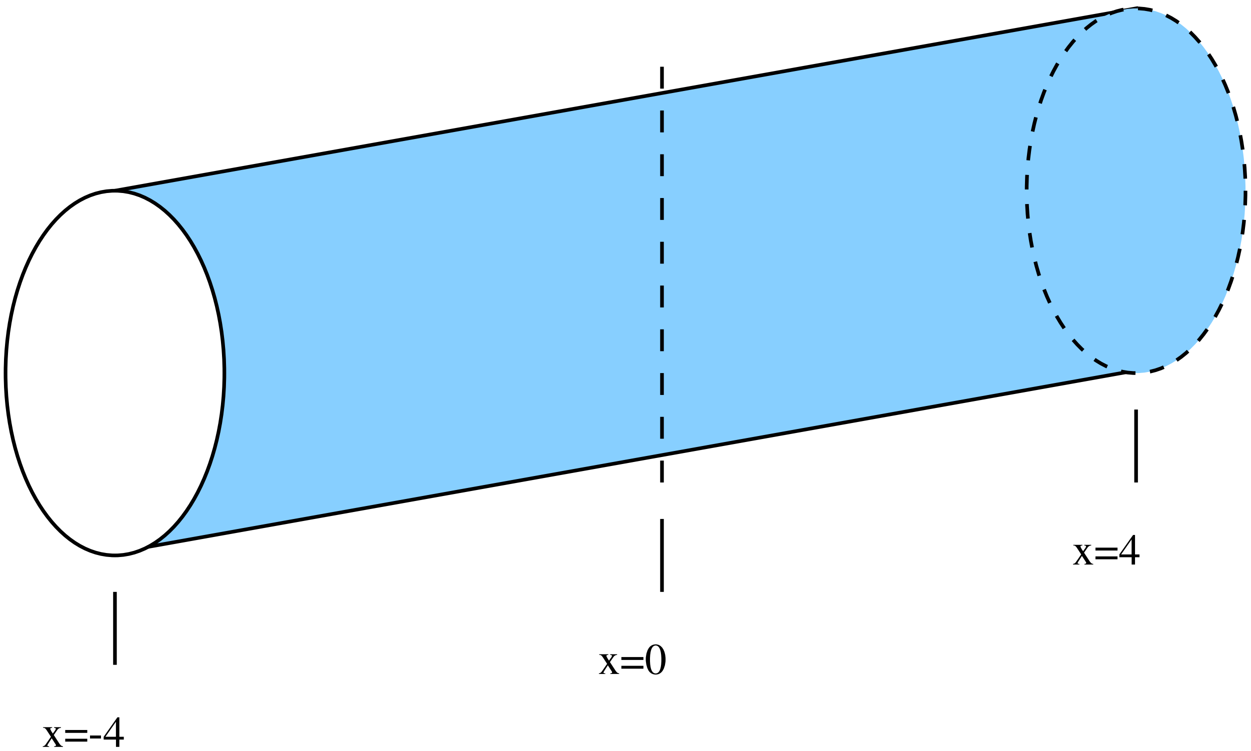

Figure 1: Shape of the wooden rod (see text).