

Problem 1. In all parts of the problem below, you can use without deriving the following solutions of the heat equation $u_t(x, t) = \alpha^2 u_{xx}(x, t)$, $x \in [0, L]$, $t \geq 0$, with appropriate boundary conditions; the first expression is for zero temperature at both boundaries (homogeneous Dirichlet BCs), and the second is for zero heat flux at both boundaries (homogeneous Neumann BCs):

$$u(x, t) = \sum_{n=1}^{\infty} b_n \exp \left\{ - \left(\frac{n\pi\alpha}{L} \right)^2 t \right\} \sin \frac{n\pi x}{L} ,$$

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp \left\{ - \left(\frac{n\pi\alpha}{L} \right)^2 t \right\} \cos \frac{n\pi x}{L} .$$

- (a) Solve the Dirichlet BVP below, find the asymptotic temperature, $u_{\infty}(x) := \lim_{t \rightarrow \infty} u(x, t)$, and explain why the expression you obtained for $u_{\infty}(x)$ is physically obvious.

$$\begin{aligned} u_t &= 9u_{xx} , & x \in [0, \pi] , & t \geq 0 , \\ u(0, t) &= 0 , & u(\pi, t) &= 0 , \\ u(x, 0) &= 4 \sin 2x + 7 \sin 5x . \end{aligned}$$

- (b) Derive and use a trigonometric relation to solve the following Dirichlet BVP:

$$\begin{aligned} u_t &= u_{xx} , & x \in [0, \pi] , & t \geq 0 , \\ u(0, t) &= 0 , & u(\pi, t) &= 0 , \\ u(x, 0) &= 4 \sin 4x \cos 2x . \end{aligned}$$

- (c) Solve the Neumann BVP below, find the asymptotic temperature, $u_{\infty}(x) := \lim_{t \rightarrow \infty} u(x, t)$, and explain why the expression you obtained for $u_{\infty}(x)$ is physically obvious.

$$\begin{aligned} u_t &= 9u_{xx} , & x \in [0, 5] , & t \geq 0 , \\ u_x(0, t) &= 0 , & u_x(5, t) &= 0 , \\ u(x, 0) &= 7 + 6 \cos 2\pi x . \end{aligned}$$

- (d) Solve the Neumann BVP below. You may use the results of Problem 4 of Section 9.3 without deriving them.

$$\begin{aligned} u_t &= 9u_{xx} , & x \in [0, 2] , & t \geq 0 , \\ u_x(0, t) &= 0 , & u_x(2, t) &= 0 , \\ u(x, 0) &= f(x) := \begin{cases} x & \text{for } x \in [0, 1] , \\ 2 - x & \text{for } x \in [1, 2] . \end{cases} \end{aligned}$$

Problem 2. Consider the following BVP with non-homogeneous Dirichlet BCs:

$$\begin{aligned} u_t &= 9u_{xx}, & x \in [0, \pi], & t \geq 0, \\ u(0, t) &= 0, & u(\pi, t) &= 5, \\ u(x, 0) &= 0. \end{aligned}$$

- (a) Set $u(x, t) = \ell(x) + v(x, t)$, where $\ell(x)$ is a linear function of x that satisfies the conditions $\ell(0) = 0$ and $\ell(\pi) = 5$ (compare these with the boundary conditions that the function u satisfies). Clearly, there is only one such linear function ℓ , namely, $\ell(x) = \frac{5}{\pi}x$. Derive the BVP satisfied by the function $v(x, t)$ – you will obtain a BVP with homogeneous (i.e., zero) Dirichlet BCs. Be careful – the PDE for v may be different than the PDE for u , and the IC for v will certainly be different from the one for u .
- (b) Solve the BVP for v derived in part (a). You again may use the expressions for the solutions of BVPs for the heat equation given in Problem 1 (without deriving them). Also, you may use the fact that the sine Fourier series of the function $f(x) = x$ for $x \in [0, L]$ is

$$\frac{2L}{\pi} \left(\sin \frac{\pi x}{L} - \frac{1}{2} \sin \frac{2\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} - \frac{1}{4} \sin \frac{4\pi x}{L} + \dots \right) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin \frac{n\pi x}{L}$$

(this expression is derived in Example 1 on pages 600–601 of the book).

- (c) Having solved part (b), write down the solution $u(x, t)$ of the original BVP.

Problem 3. In this problem you can use without deriving that the solution of the wave equation $u_{xx}(x, t) - \frac{1}{c^2}u_{tt}(x, t) = 0$, $x \in [0, L]$, $t \geq 0$, with homogeneous Dirichlet BCs $u(0, t) = 0$, $u(L, t) = 0$ has the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L}.$$

The functions $T_n(t)$ satisfy the ODEs

$$T_n''(t) + \left(\frac{n\pi c}{L} \right)^2 T_n(t) = 0, \quad t \geq 0,$$

so their general form is

$$T_n(t) = A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}.$$

The constants A_n and B_n can be found from the initial conditions, $u(x, 0) = f(x)$ (initial position of the spring) and $u_t(x, 0) = g(x)$ (initial velocity of the spring).

Solve the BVP

$$\begin{aligned}u_{xx} - \frac{1}{9}u_{tt} &= 0, & x \in [0, \pi], & \quad t \geq 0, \\u(0, t) &= 0, & u(\pi, t) &= 0, \\u(x, 0) &= 4 \sin 2x, & u_t(x, 0) &= 15 \sin 5x.\end{aligned}$$