Math 3413.001: Physical Mathematics I

Homework 10, due April 14 (Tuesday)

Lecture 21 (Mar 31) Due date 04/14/2020 : Section 9.3/9.5

1. Find a formal series solution for the following differential equation.

$$x'' + 2x = f(t), \qquad x(0) = 0 = x(2), \qquad f(t) = \begin{cases} 1 & \text{if } 0 < t < 1; \\ 0 & \text{if } 1 \le t < 2. \end{cases}$$

2. Find a formal series solution for the following differential equation.

$$x'' + 2x = f(t), \qquad x'(0) = 0 = x'(2), \qquad f(t) = \begin{cases} 1 & \text{if } 0 < t < 1; \\ 0 & \text{if } 1 \le t < 2. \end{cases}$$

3. The goal is to solve the boundary value problem

$$u_t = 2u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = 31\sin(2\pi x) + 5\sin(3\pi x).$$

   (a) Show that the following functions $u_1(x,t), u_2(x,t), u_3(x,t)$ satisfy $u_t = 2u_{xx}$ and $u(0,t) = u(1,t) = 0$.

$$u_1(x,t) = e^{-2\pi^2 t}\sin(\pi x), \quad u_2(x,t) = e^{-8\pi^2 t}\sin(2\pi x), \quad u_1(x,t) = e^{-18\pi^2 t}\sin(3\pi x).$$

   (b) Find constants c_1, c_2, c_3 such that $u(x,t) = c_1u_1(x,t) + c_2u_2(x,t) + c_3u_3(x,t)$ is a solution to the above boundary value problem.

Suggested problems from the book (DO NOT SUBMIT): Pg 589-590, #12, 14

Lecture 22 (Apr 2) Due date 04/14/2020 : Section 9.5

1. Solve the following boundary value problem

$$u_t = 5u_{xx}, \qquad u(0,t) = u(2,t) = 0, \qquad u(x,0) = f(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{if } 1 \le x < 2. \end{cases}$$

2. Solve the following boundary value problem

$$4u_t = u_{xx}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = \sin(x)\big(5 - 12\cos(x)\big)$$

3. Consider the temperature $u(x,t)$ of a slender wire with $u(0,t) = u(L,t) = 0$ and $u(x,0) = f(x)$. Instead of being laterally insulated, the wire loses heat to a surrounding medium (at fixed temperature zero) at a rate proportional to $u(x,t)$. It turns out that $u(x,t)$ satisfies the partial differential equation

$$u_t = ku_{xx} - hu, h \text{ is a positive constant}, \quad u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x). \qquad (*)$$

   (a) Suppose $u(x,t)$ is a solution of the above PDE. Set $u(x,t) = e^{-ht}v(x,t)$. Then show that $v(x,t)$ satisfies the boundary value problem

$$u_t = ku_{xx}, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x).$$

(b) Use the solution to the BVP from the lecture notes to conclude that the solution to the PDE $(*)$ above is given by

$$u(x,t) = e^{-ht} \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2}{L^2}tk} \sin(\frac{n\pi x}{L}), \quad B_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx.$$

Suggested problems from the book (DO NOT SUBMIT): Pg 608-610, $\#1, 3, 10, 13$