

Problem 1. Use the relations $\mathbf{e}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ and $\mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$ (which are obvious from the definitions of \mathbf{e}_r and \mathbf{e}_θ) to show that $\frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r$.

Problem 2. Use the relations $\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta$ and $\frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r$ to determine $\frac{d\mathbf{u}}{dt}$ if

$$\mathbf{u}(t) = \sin \theta(t) \mathbf{e}_r - r^2(t) \theta(t) \mathbf{e}_\theta, \quad \text{where } r(t) = t^2, \quad \theta(t) = 2t.$$

Problem 3.

(a) Use the relations

$$\mathbf{e}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad \mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$$

to express \mathbf{i} and \mathbf{j} in terms of \mathbf{e}_r and \mathbf{e}_θ .

(b) Let $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j}$. Find $\mathbf{v}(t) := \mathbf{r}'(t)$ in Cartesian coordinates.

(c) Use the identity $1 + \tan^2 \theta = \frac{1}{\cos^2 \theta}$ to express $\cos \theta$ and $\sin \theta$ as functions of $\tan \theta$ only. Assume that $\theta \in [0, \frac{\pi}{2})$.

(d) Use your results from parts (a), (b) and (c) to express $\mathbf{v}(t)$ in polar coordinates for $t > 0$. You will need to express $\sin \theta(t)$ and $\cos \theta(t)$ as functions of t only. To this end, you can use that, for $t > 0$, $\theta(t) = \arctan \frac{y(t)}{x(t)} = \arctan \frac{t^2}{t} = \arctan t$.

Problem 4.

(a) Use equation (25) on page 359 of the book to find the divergence of the vector field $\mathbf{u} = r \cos \theta \mathbf{e}_r - r \sin \theta \mathbf{e}_\theta$.

(b) Express the vector field \mathbf{u} in Cartesian coordinates. You will need the result of Problem 3(a).

(c) Find the divergence of \mathbf{u} in Cartesian coordinates. Does your result agree with what you found in part (a)?

Problem 5. In this problem you will prove some properties of a particular type of orthogonal coordinate system in \mathbf{R}^2 , the *parabolic coordinates*. The parabolic coordinates (σ, τ) are related to the Cartesian coordinates (x, y) by

$$x = \sigma\tau \tag{1}$$

$$y = \frac{1}{2}(\tau^2 - \sigma^2), \tag{2}$$

where $\sigma \in \mathbb{R}$, $\tau \in [0, \infty)$.

- (a) Expressing $\tau = \frac{x}{\sigma}$ from (1) and substituting it in (2), we see that the surfaces (in the case of two dimensions, lines) $\sigma = \text{const}$ are the parabolas

$$y = \frac{x^2}{2\sigma^2} - \frac{\sigma^2}{2}.$$

Find the equation of the lines $\tau = \text{const}$. Identify the families $\sigma = \text{const}$ and $\tau = \text{const}$ in Figure 1.

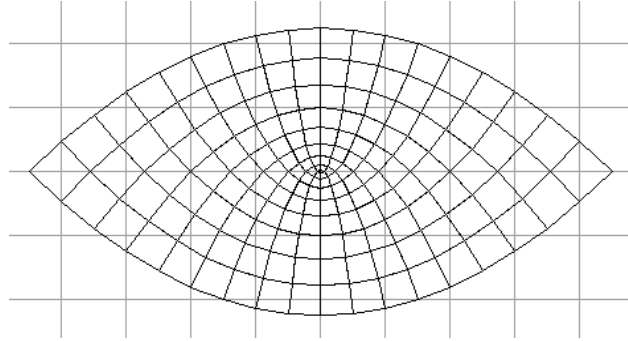


Figure 1: Coordinate lines $\sigma = \text{const}$ and $\tau = \text{const}$.

- (b) Let \mathbf{e}_σ and \mathbf{e}_τ be the unit vectors in the coordinates (σ, τ) . If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, use (1) and (2) to show that

$$\frac{\partial \mathbf{r}}{\partial \sigma} = \tau \mathbf{i} - \sigma \mathbf{j} \quad (3)$$

$$\frac{\partial \mathbf{r}}{\partial \tau} = \sigma \mathbf{i} + \tau \mathbf{j}. \quad (4)$$

- (c) The *metric tensor* (g_{ij}) in coordinates (u_1, \dots, u_n) is given by

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j}.$$

In this problem, it is

$$(g_{ij}) = \begin{pmatrix} g_{\sigma\sigma} & g_{\sigma\tau} \\ g_{\tau\sigma} & g_{\tau\tau} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathbf{r}}{\partial \sigma} \cdot \frac{\partial \mathbf{r}}{\partial \sigma} & \frac{\partial \mathbf{r}}{\partial \sigma} \cdot \frac{\partial \mathbf{r}}{\partial \tau} \\ \frac{\partial \mathbf{r}}{\partial \tau} \cdot \frac{\partial \mathbf{r}}{\partial \sigma} & \frac{\partial \mathbf{r}}{\partial \tau} \cdot \frac{\partial \mathbf{r}}{\partial \tau} \end{pmatrix}.$$

Find the explicit expression of the metric tensor in parabolic coordinates.

- (d) From the components of the metric tensor, one can immediately see that (σ, τ) are orthogonal coordinates. How?

- (e) Find the *metric coefficients* h_σ and h_τ .

Hint: The metric coefficients for an orthogonal coordinate system are defined by equation (13) on page 357 of the book. However, you have already done all the calculations – the metric coefficients are directly related to the components of the metric tensor.

- (f) If the metric tensor in coordinates (u_1, \dots, u_n) is (g_{ij}) , the n -dimensional volume element in these coordinates is given by

$$dV = \sqrt{\det(g_{ij})} du_1 \cdots du_n .$$

In two dimensions, the “volume” is the area, so that the area element dA in coordinates (u_1, u_2) is given by

$$dA = \sqrt{\det(g_{ij})} du_1 du_2 . \quad (5)$$

Show that, in parabolic coordinates (σ, τ) , this equation becomes

$$dA = (\sigma^2 + \tau^2) d\sigma d\tau .$$

- (g) Now consider the domain \mathcal{D} in the (x, y) plane between the lines $x = 0$, $y = 0$, and $y = 1 - \frac{x^2}{4}$, shown in Figure 2. Find the area A of this domain by using each of the

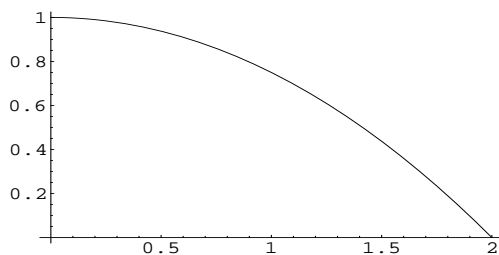


Figure 2: The domain \mathcal{D} in the (x, y) plane.

following three methods (in all these methods use Cartesian coordinates); clearly, each method should give you the same number.

- Use the formula

$$A = \iint dx dy ,$$

where the integration is over the domain

$$\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, y \leq 1 - \frac{x^2}{4} \right\} .$$

Note that this formula for the area is a particular case of (5) because the metric tensor in Cartesian coordinates is the unit matrix $(g_{ij} = \delta_{ij})$, hence $\det(g_{ij}) = 1$.

- Compute the area A as the area under the graph of the curve $y = Y(x) := 1 - \frac{x^2}{4}$:

$$A = \int_0^2 \left(1 - \frac{x^2}{4}\right) dx .$$

- Now consider y as an independent variable, and $x = X(y)$ as a function, and compute the area A as

$$A = \int_0^? X(y) dy .$$

(What is the upper limit of integration?)

- (h) Now draw the domain \mathcal{D} defined above in the (σ, τ) plane. For your convenience, the domain \mathcal{D} is drawn in coordinates (σ, τ) in the figure below (σ is on the horizontal axis, τ is on the vertical axis). The segment of the y -axis between the points $(x, y) = (0, 0)$

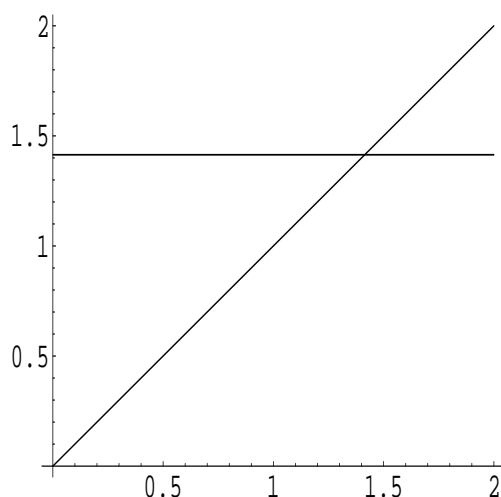


Figure 3: The domain \mathcal{D} in the (σ, τ) plane.

and $(x, y) = (0, 1)$ in Figure 2, is again a straight-line segment in the (σ, τ) plane (Figure 3), connecting the points $(\sigma, \tau) = (0, 0)$ and $(\sigma, \tau) = (0, \sqrt{2})$. You have to write down the equations of the other two parts of the boundary of \mathcal{D} in Figure 2 and to indicate them in Figure 3.

Hint: The parabola in Figure 2 is one of the lines $\tau = \text{const}$; what is the value of the constant?

- (i) Find the area of the domain \mathcal{D} as seen in coordinates (σ, τ) , i.e., compute

$$\iint (\sigma^2 + \tau^2) d\sigma d\tau ,$$

where the integration is over the domain drawn in Figure 3. You have already computed this area using coordinates (x, y) , but I want to see your detailed computations in coordinates (σ, τ) .

- (j) Follow the ideas from Section 8.2 of the book to find an expression for the gradient of a scalar function $f(\sigma, \tau)$ in parabolic coordinates. Namely, write

$$d\mathbf{r} = h_\sigma d\sigma \mathbf{e}_\sigma + h_\tau d\tau \mathbf{e}_\tau, \quad (6)$$

$$\nabla f = (\nabla f)_\sigma \mathbf{e}_\sigma + (\nabla f)_\tau \mathbf{e}_\tau,$$

and use that

$$df = (\nabla f) \cdot d\mathbf{r} = \frac{\partial f}{\partial \sigma} d\sigma + \frac{\partial f}{\partial \tau} d\tau$$

to find expressions for ∇f and the differential operator ∇ (similarly to the equations (22) and (23) on page 359 of the book).

- (k) Finally, you have to compute the length L of the straight-line segment connecting the points $(x, y) = (0, 0)$ and $(x, y) = (2, 0)$ in Figure 2, but using coordinates (σ, τ) . Clearly, you have to obtain that $L = 2$. To find L , identify the line segment in the (σ, τ) plane (i.e., in Figure 3), and parametrize it, i.e., write the points in it in the form $(\sigma, \tau) = (\Sigma(t), \mathbf{T}(t))$, where the real parameter t . Since the length element $ds = |d\mathbf{r}|$ is given by

$$ds = \sqrt{h_\sigma^2 d\sigma^2 + h_\tau^2 d\tau^2}$$

(which is easily obtained by squaring (6)), the length L can be obtained as

$$L = \int ds = \int_{?}^{?} \left[h_\sigma^2 \left(\frac{d\Sigma}{dt} \right)^2 + h_\tau^2 \left(\frac{d\mathbf{T}}{dt} \right)^2 \right]^{1/2} dt$$

(where you have to figure out the limits of integration; clearly, they will depend on the way you parametrized the line segment).