# MATH 4073 Homework 10 Due Wed, Nov 9

**Problem 1.** The Trapezoidal rule applied to $\int_0^2 f(x)\,\mathrm{d}x$ gives the value 5, and the Midpoint rule gives the value 4. What value does Simpson's rule give?

**Problem 2.** The quadrature formula

$$\int_0^2 f(x)\,\mathrm{d}x = c_0 f(0) + c_1 f(1) + c_2 f(2)$$

is exact for all polynomials of degree less than or equal to 2. Determine $c_0$, $c_1$, and $c_2$.

**Problem 3.** Determine the values of $n$ and $h$ required to approximate

$$\int_1^2 x \ln x \,\mathrm{d}x$$

to within $10^{-5}$ and compute the approximation. Use:

(a) the Composite Trapezoidal rule;

(b) the Composite Simpson's rule;

(c) the Composite Midpoint rule.

**Problem 4.** Trying to compute the numerical values of integrals with infinite limits is often complicated, so one needs to apply some alternative technique. In this problem you will estimate the numerical value of the integral

$$\int_b^\infty \frac{\mathrm{d}x}{1+\mathrm{e}^{-x}+x^2}\ ,$$

where $b > 0$ is a constant, by proving rigorous lower and upper bounds for its value.

Prove that, for any positive constant $b$,

$$\frac{1}{\sqrt{1+\mathrm{e}^{-b}}}\left(\frac{\pi}{2} - \arctan\frac{b}{\sqrt{1+\mathrm{e}^{-b}}}\right) < \int_b^\infty \frac{\mathrm{d}x}{1+\mathrm{e}^{-x}+x^2} < \frac{\pi}{2} - \arctan b\ .$$

Take $b = 5$ and find the numerical values of the bounds. How many correct digits of the numerical value of the integral do the bounds provide for this value of $b$?

**Problem 5.** The antiderivative of the function $\mathrm{e}^{-x^2}$ cannot be written in terms of elementary functions, but is is known that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\ .$$

The direct numerical computation of this integral is not easy because of the infinite limits of integration. Here you will compute the numerical value of this integral by using two methods to avoid dealing with infinite limits. After performing some preliminary work you will need to compute the numerical values with the MATLAB programs `traprl.m` (Trapezoidal rule) or `simprl.m` (Simpson's rule) available at the class web-site; these programs are taken from the book *Numerical Methods Using MATLAB* by J. H. Mathews and K. D. Fink (4th edition, Pearson Prentice Hall, 2004).

(a) Change variables in the integral to make the limits of integration finite. Compute the numerical value of the transformed integral using MATLAB.

*Hint:* The function tan is a one-to-one mapping from the finite interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ to the whole real line.

(b) The function $\mathrm{e}^{-x^2}$ decreases quite fast as the distance from $x$ to 0 increases. Show that if, instead of $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x$ we compute $2\int_0^5 \mathrm{e}^{-x^2}\,\mathrm{d}x$, the error will not exceed $\frac{2}{5}\,\mathrm{e}^{-25}$. Compute the numerical value of the transformed integral using MATLAB.

*Hint:* To give an upper bound on the value of the "tail" $\int_5^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x$, replace the integrand with some function which also decreases very fast, but whose antiderivative you know.

**Problem 6.** The integral

$$\int_0^3 \frac{\sin x}{x^{3/2}}\,\mathrm{d}x$$

is perfectly well defined, and its value is $2.64958748923\ldots$. However, if you attempt to integrate it directly with the programs `traprl.m` or `simprl.m`, MATLAB will not do it (and will complain about division by zero). In fact, the integrand does become infinite at the origin, but nevertheless the integral of it is finite. To help MATLAB, decompose the integral as

$$\int_0^3 \frac{\sin x}{x^{3/2}}\,\mathrm{d}x = \int_0^{0.1} \frac{\sin x}{x^{3/2}}\,\mathrm{d}x + \int_{0.1}^3 \frac{\sin x}{x^{3/2}}\,\mathrm{d}x\ ,$$

and compute the integral from 0.1 to 3 with MATLAB (this time MATLAB won't mind).

To compute the numerical value of the integral from 0 to 0.1, expand $\sin x$ in a Taylor series around 0 (keep no more than 3-4 terms), then divide by $x^{3/2}$, and integrate term by term. Give an upper bound on the error in the value of the integral from 0 to 0.1 computed in this way if you keep only the linear and the cubic term in the Taylor expansion of $\sin x$.

*Hint:* When you truncate an alternating series, the error does not exceed the first neglected term.