MATH 4433 Homework 10 Due 11/13/2014

Section 5.2: Exercises 4, 8. Hints and remarks:

- in Exercise 4, use directly the definition of continuity, and notice that $f(x) - f(3) = (x-3)(x+6)$;
- in Exercise 8, you are allowed to use all theorems you know about limits (i.e., work as if this were a problem in Calculus), in particular, that $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$; you have to study separately the cases $|x| > 1$, $|x| < 1$, $x = -1$, and $x = 1$; after you found f, plot a graph of it.

Section 5.3: Exercises 3(a,b,c,d,e,f,h), 4, 7, 9. Hints and remarks:

- the answers for Exercise 3 are given in the back of the book; do not confuse "finite" with "bounded", and "infinite" with "unbounded"; in part (b) the set D cannot be bounded (because of Theorem 5.3.2 and Heine-Borel Theorem), so take $D = [0, \infty)$ and $f : D \to \mathbb{R}$ defined by $f(x) = 1 - \mathrm{e}^{-x}$; in part (c) consider, say, $f(x) = x^2$ and take D to be the union of an open interval and a one-point set; one possible function in part (d) is the inverse of the function from part (b) (with appropriately defined domain), or take the function $f(x) = x^2$ with $D = (-1, 2]$;
- there is a hint in the book for Exercise 7.

Section 6.1: Exercises 3(a,b), 4(c,d), 5, 9, 11, 18. Hints and remarks:

- in Exercise 3(a), you have to find separately the left-handed and the right-handed limits of $\dfrac{f(x) - f(1)}{x - 1}$; recall the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$;
- you have to solve Exercises 4(c,d) directly from the definition of derivative; in Exercise 4(d), a useful identity is $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$;
- in Exercise 5(a), use the identity $a^3 - b^3 = (\sqrt[3]{a} - \sqrt[3]{b})(a^{2/3} + \sqrt[3]{ab} + b^{2/3})$;
- in Exercise 11(b), you have to prove only that f is differentiable at $x = 0$ (why?), which can be done very easily by using directly the definition of derivative;

Please turn the page!

Food for Thought:

- Sec. **5.3**, exercises 1, 2, 12 (Exercise 12 is more difficult, but it would be a great exercise for you; relevant theorems that you will need in the proof are Heine-Borel Theorem, Bolzano-Weierstrass Theorem for sequences, and the result of Exercise 4.1.15; the easiest way to prove the boundedness of $f(D)$ is by contradiction).

- Sec. **6.1**, exercises 1, 2, 7.

Solution of Exercise 5.1.12.

Method 1. The easiest way to show the continuity of $\sqrt{f}$ on D is to use the fact that $\sqrt{f}$ can be written as a composition of two functions: $\sqrt{f} = g \circ f$, where $g : [0, \infty) \to \mathbb{R}$ is defined as $g(x) = \sqrt{x}$. The continuity of $\sqrt{f}$ follows directly from Theorem 5.2.12.

Method 2. One can use directly the definition of continuity. We will consider the cases $f(c) = 0$ and $f(c) > 0$ separately.

Case 1: $f(c) = 0$. Let $\varepsilon > 0$ be an arbitrary positive number. We have to find $\delta > 0$ such that if $x \in D$ satisfies $|x - c| < \delta$, then $\left|\sqrt{f(x)} - \sqrt{f(c)}\right| < \varepsilon$. Since f is continuous at $c = 0$ and $\varepsilon^2 > 0$, we can find $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon^2 \qquad \forall\, x \in N(c, \delta) \cap D\ .$$

Since we assumed that $f(c) = 0$ and we know that $f(x) \geq 0\ \forall x \in D$, we have $|f(x) - f(c)| = |f(x)| = f(x)$. Using this, for the δ chosen above, we have

$$\left|\sqrt{f(x)} - \sqrt{f(c)}\right| = \left|\sqrt{f(x)} - \sqrt{0}\right| = \sqrt{f(x)} < \sqrt{\varepsilon^2} < \varepsilon \qquad \forall\, x \in N(c, \delta) \cap D\ .$$

Case : $f(c) > 0$. First note that the continuity of f at c guarantees that, for every $\alpha > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \alpha \qquad \forall\, x \in N(c, \delta) \cap D\ ; \tag{1}$$

we will chose the value of α below. This, in particular, allows us to take δ small enough so that $f(x)$ is strictly positive $\forall\, x \in N(c, \delta) \cap D$ – to achieve this, simply take, say, $\alpha = \frac{1}{5} f(c) > 0$ in (1), and let $\delta_1 > 0$ be the corresponding value of δ, then $f(x) > \frac{4}{5} f(c) > 0\ \forall\, x \in N(c, \delta_1) \cap D$ (why)?

Let $\varepsilon > 0$ be an arbitrary positive number. For any $0 < \delta \leq \delta_1$, we have

$$\left|\sqrt{f(x)} - \sqrt{f(c)}\right| = \frac{|f(x) - f(c)|}{\sqrt{f(x)} + \sqrt{f(c)}} < \frac{|f(x) - f(c)|}{\sqrt{f(c)}} < \frac{\alpha}{\sqrt{f(c)}} \overset{(?)}{<} \varepsilon\ .$$

To satisfy the inequality $\overset{(?)}{<}$, we choose $\alpha = \frac{1}{2} \varepsilon \sqrt{f(c)} > 0$ in (1); let $\delta_2 > 0$ be the value of δ that makes this happen. Choosing $\delta := \min\{\delta_1, \delta_2\}$, we have achieved that $\left|\sqrt{f(x)} - \sqrt{f(c)}\right| < \varepsilon$ for every $x \in N(c, \delta) \cap D$.